

Solutions: Homework 1

Ex. 1.1: Capacitor

- (a) (*This should be a familiar exercise, so I will not include diagrams.*) First consider an infinite plane, with uniform surface charge density ρ_s . Applying Gauss's law, with a Gaussian surface of uniform cross-section, having area A , we have

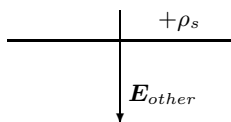
$$2EA = \oint \mathbf{E} \cdot d\mathbf{A} = 4\pi \int \rho_s dA = 4\pi \rho_s A ,$$

$$\Rightarrow E = 2\pi \rho_s . \quad (1)$$

Ignoring edge effects, our capacitor is equivalent to two infinite parallel planes, with surface charges of opposite sign. Then from (1), the superposition principle immediately implies that the electric field between the plates has magnitude

$$E = 4\pi \rho_s . \quad (2)$$

- (b) Consider one infinite sheet in our capacitor. The view looking edge-on to this sheet is shown below. The force on the sheet is due to the Lorentz interaction with the electric field of the other plate,



$\mathbf{F} = q\mathbf{E}_{other}$, where q is the charge on the sheet and $E_{other} = 2\pi\rho_s$ from above. By definition the pressure is $p = dF/dA$, and note that also by definition $\rho_s = dq/dA$. Since the electric field is constant, then

$$p = \frac{dq}{dA} E_{other} = 2\pi \rho_s^2 . \quad (3)$$

Note this pressure is attractive.

- (c) Including dielectrics polarizations, Gauss's law in integral form becomes

$$\oint \mathbf{D} \cdot d\mathbf{A} = 4\pi q_f , \quad (4)$$

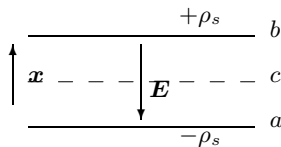
where q_f are the free charges. The \mathbf{D} field depends only on free charges, which in this system lie on the two sheets forming the capacitor with density $\pm\rho_s$: there are no free charges inside or on the dielectric. Hence, similarly to part (a) we find $D = 4\pi\rho_s$ between the plates, both inside and outside the dielectric.

Now, outside the dielectric in the vacuum, $D = E$, so

$$E_{out} = 4\pi \rho_s . \quad (5)$$

The dielectric material has dielectric constant ε , and by definition $D = \varepsilon E$. So inside the dielectric

$$E_{in} = \frac{1}{\varepsilon} 4\pi \rho_s . \quad (6)$$



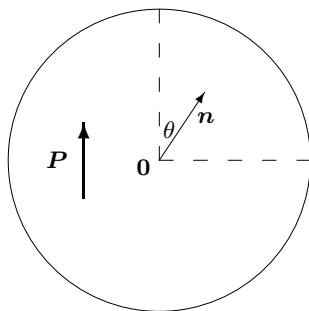
- (d) Remember that to find the potential, you should carefully define a coordinate system, shown below. Then potential at b with respect to a is

$$\begin{aligned}
 V_{ba} &= - \int_a^b \mathbf{E} \cdot d\mathbf{x} \\
 &= + \int_a^c E_{in} dx + \int_c^b E_{out} dx \\
 &= 4\pi\rho_s d/2 + (1/\epsilon)4\pi\rho_s d/2 \\
 &= 2\pi\rho_s d(1/\epsilon + 1) ,
 \end{aligned} \tag{7}$$

which is positive, as expected.

Ex. 1.2: Spherical Cavity

- (a) Consider a sphere S of radius R with uniform polarization $\mathbf{P} = P_0 \mathbf{e}_z$, shown below. First, the



polarization is uniform so that clearly $\nabla \cdot \mathbf{P} = 0$, and hence the volumetric bound charge density $\rho_b = 0$. The surface bound charge density $\rho_{sb} = \mathbf{n} \cdot \mathbf{P}$ is nonzero, however.

Now, in the spherical coordinates (r, ϕ, θ) , where ϕ and θ are the azimuthal and inclination angles respectively, the unit normal vector is

$$\mathbf{n} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) , \tag{8}$$

and so

$$\rho_{sb} = P_0 \cos \theta . \tag{9}$$

Coulomb's law then tells us that the field at the origin due to the charge on an differential area dA is simply

$$d\mathbf{E}(\mathbf{0}) = -\mathbf{n} \frac{\rho_{sb}}{R^2} dA = -\mathbf{n} P_0 \cos \theta d\Omega , \tag{10}$$

since $dA = R^2 d\Omega$ on the surface of a sphere and we have used (9). Note the field points inwards for positive ρ_s whence the minus sign. Applying the principle of superposition, we integrate over

the spherical surface S , so that

$$\begin{aligned}
 \mathbf{E}(\mathbf{0}) &= -P_0 \int_0^{2\pi} d\phi \int_{-1}^1 d(\cos \theta) \left[\cos \theta (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) \right] \\
 &= -2\pi P_0 \mathbf{e}_z \int_{-1}^1 d(\cos \theta) \cos^2 \theta \\
 &= -\frac{4\pi}{3} P_0 \mathbf{e}_z = -\frac{4\pi}{3} \mathbf{P} .
 \end{aligned} \tag{11}$$

- (b) Consider two spheres labelled 1 and 2, with respective centers located at $\delta/2$ and $-\delta/2$ and respective uniform charge densities $+\rho_0$ and $-\rho_0$. Outside each sphere, the electric field due the sphere in question looks like that of a point charge, with charge $q = \pm\rho_0 V$. The dipole moment is then clearly just $q\delta$, so that the dipole moment per unit volume - the polarization - is

$$\mathbf{P} = \rho_0 \delta . \tag{12}$$

Hence we need $P_0 = \rho_0 \delta$ for these two systems to have the same polarization.

The electric field inside sphere 1 is

$$\mathbf{E}_1(\mathbf{r}) = \frac{4\pi\rho_0}{3} (\mathbf{r} - \delta/2) , \tag{13}$$

and for sphere 2

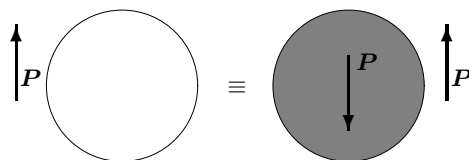
$$\mathbf{E}_2(\mathbf{r}) = -\frac{4\pi\rho_0}{3} (\mathbf{r} + \delta/2) , \tag{14}$$

We assume that $\delta \ll R$, so that the overlap of the spheres is approximately complete, so the intersection of the spheres is approximately our original uniformly polarized sphere. The field inside this sphere is then, by superposition

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_1 + \mathbf{E}_2 = -\frac{4\pi\rho_0}{3} \delta , \tag{15}$$

which is a constant.

- (c) Finally, consider a spherical cavity in a polarized medium, with uniform polarization \mathbf{P} . The spherical cavity is equivalent to superimposing a sphere of polarization $-\mathbf{P}$ on the medium, as shown below. From (11), the field in the cavity due to the $-\mathbf{P}$ polarization is $\mathbf{E}_{pol} = (4\pi/3)\mathbf{P}$. Hence, if the field in

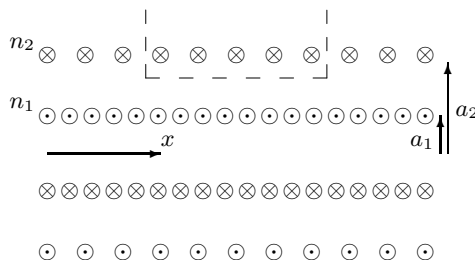


the polarized material is \mathbf{E} , then by superposition the field in the cavity is

$$\mathbf{E}_{cav} = \mathbf{E} + \frac{4\pi}{3} \mathbf{P} . \tag{16}$$

Ex 1.3: Coaxial Solenoids

(Again, this exercise should be familiar, so I'll spare some details.) Consider two infinitely long coaxial solenoids, shown below in sectional view. We assume the solenoids are wound with the same orientation. For each solenoid, applying Ampère's law using a loop of the form indicated by the dashed line, having finite



side length L (the other sides extend off to infinity), one finds

$$-B_2 L = \oint \mathbf{B} \cdot d\mathbf{x} = (4\pi/c) I_{\text{enc}} = (4\pi/c) n_2 L I . \quad (17)$$

Hence the field due to the outer solenoid is $\mathbf{B}_2 = -(4\pi/c) n_2 I \mathbf{e}_x$ inside the solenoid, and one can deduce the field outside is zero by using a similar rectangular loop with only finite sides.

Applying the superposition principle, and noting that the currents flow in different directions in each solenoid

(a)

$$\mathbf{B}(r < a_1) = (4\pi/c) I (n_1 - n_2) \mathbf{e}_x . \quad (18)$$

(b)

$$\mathbf{B}(a_1 < r < a_2) = -(4\pi/c) I n_2 \mathbf{e}_x . \quad (19)$$

(c)

$$\mathbf{B}(r > a_2) = \mathbf{0} . \quad (20)$$

Ex 1.4: Newton's 3rd Law

Consider two current elements $I_1 d\mathbf{l}_1$ and $I_2 d\mathbf{l}_2$, which are located at \mathbf{l}_1 and \mathbf{l}_2 by definition and which belong to circuits Γ_1 and Γ_2 respectively. Applying the Biot-Savart Law, the magnetic field due to 1 at 2 and due to 2 at 1 is respectively

$$\begin{aligned} d\mathbf{B}_1(\mathbf{l}_2) &= \frac{1}{c} I_1 d\mathbf{l}_1 \times \frac{(\mathbf{l}_2 - \mathbf{l}_1)}{\|\mathbf{l}_2 - \mathbf{l}_1\|^3} \\ d\mathbf{B}_2(\mathbf{l}_1) &= \frac{1}{c} I_2 d\mathbf{l}_2 \times \frac{(\mathbf{l}_1 - \mathbf{l}_2)}{\|\mathbf{l}_2 - \mathbf{l}_1\|^3} . \end{aligned} \quad (21)$$

The Lorentz force of 1 on 2 is then

$$d\mathbf{F}_{12} = \frac{I_1 I_2}{c} d\mathbf{l}_2 \times \left[d\mathbf{l}_1 \times \frac{(\mathbf{l}_2 - \mathbf{l}_1)}{\|\mathbf{l}_2 - \mathbf{l}_1\|^3} \right] , \quad (22)$$

where the square brackets are needed as the triple cross-product is not associative, and $d\mathbf{F}_{21}$ is found by exchanging the subscripts everywhere. Applying the Lagrange expansion of the triple cross-product, we may write the differential force as

$$d\mathbf{F}_{12} = \frac{I_1 I_2}{c} \left[d\mathbf{l}_1 \left(d\mathbf{l}_2 \cdot \frac{(\mathbf{l}_2 - \mathbf{l}_1)}{\|\mathbf{l}_2 - \mathbf{l}_1\|^3} \right) - \frac{(\mathbf{l}_2 - \mathbf{l}_1)}{\|\mathbf{l}_2 - \mathbf{l}_1\|^3} (d\mathbf{l}_2 \cdot d\mathbf{l}_1) \right] . \quad (23)$$

In this form it is clear that $d\mathbf{F}_{12} \neq -d\mathbf{F}_{21}$ since although the second term clearly just changes sign under exchange of subscripts, the first term is a different differential form altogether.

In integral form, we have

$$\mathbf{F}_{12} = \frac{I_1 I_2}{c} \left[\oint_{\Gamma_1} d\mathbf{l}_1 \left(\oint_{\Gamma_2} d\mathbf{l}_2 \cdot \frac{\mathbf{e}_{(\mathbf{l}_2 - \mathbf{l}_1)}}{\|\mathbf{l}_2 - \mathbf{l}_1\|^2} \right) - \oint_{\Gamma_2} d\mathbf{l}_2 \cdot \oint_{\Gamma_1} d\mathbf{l}_1 \frac{\mathbf{e}_{(\mathbf{l}_2 - \mathbf{l}_1)}}{\|\mathbf{l}_2 - \mathbf{l}_1\|^2} \right]. \quad (24)$$

Note \mathbf{l}_1 and \mathbf{l}_2 are now dummy variables: to find \mathbf{F}_{21} we instead exchange the integral domains Γ_1 and Γ_2 . For convenience, the vector terms have also been rewritten in terms of the unit vector $\mathbf{e}_{(\mathbf{l}_2 - \mathbf{l}_1)} = -\mathbf{e}_{(\mathbf{l}_1 - \mathbf{l}_2)}$.

Now, in the first term we have a closed path integral over the conservative field \mathbf{e}_r/r^2 . That is

$$\oint_{\Gamma_2} d\mathbf{l}_2 \cdot \frac{\mathbf{e}_{(\mathbf{l}_2 - \mathbf{l}_1)}}{\|\mathbf{l}_2 - \mathbf{l}_1\|^2} = 0, \quad (25)$$

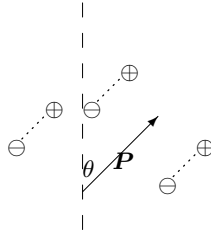
for any \mathbf{l}_1 . Hence we have

$$\begin{aligned} \mathbf{F}_{12} &= -\frac{I_1 I_2}{c} \oint_{\Gamma_2} d\mathbf{l}_2 \cdot \oint_{\Gamma_1} d\mathbf{l}_1 \frac{\mathbf{e}_{(\mathbf{l}_2 - \mathbf{l}_1)}}{\|\mathbf{l}_2 - \mathbf{l}_1\|^2} \\ &= -\frac{I_1 I_2}{c} \oint_{\Gamma_2} d\mathbf{l}_1 \cdot \oint_{\Gamma_1} d\mathbf{l}_2 \frac{\mathbf{e}_{(\mathbf{l}_1 - \mathbf{l}_2)}}{\|\mathbf{l}_1 - \mathbf{l}_2\|^2} \quad (\text{swap variables}) \\ &= +\frac{I_1 I_2}{c} \oint_{\Gamma_1} d\mathbf{l}_2 \cdot \oint_{\Gamma_2} d\mathbf{l}_1 \frac{\mathbf{e}_{(\mathbf{l}_2 - \mathbf{l}_1)}}{\|\mathbf{l}_2 - \mathbf{l}_1\|^2} \quad (\text{swap integrals}) \\ &= -\mathbf{F}_{21}, \end{aligned} \quad (26)$$

as expected.

Ex 1.5: Dipoles

- a) Consider a polarized dielectric, polarization \mathbf{P} . Suppose we coarse-grain the polarization into a set of ‘local averaging cells’ (formally this means we replace \mathbf{P} with a simple function, whose value in any cell coincides with the average of \mathbf{P} in that cell): We can consider each cell as an ensemble of dipoles, number density N , charge q and length l , that are aligned in the coarse-grained \mathbf{P} direction. Suppose a smooth surface is embedded in this dielectric. We suppose the local averaging cell scale is sufficiently small, such that the patch of this surface lying within any ‘local averaging cell’ is locally flat, as shown. Any dipole whose centre lies within $l/2 \cos \theta$ of the surface patch intersects the surface, so dipoles lying



in the elemental volume $l \cos \theta dA$ are cut. Hence the number of cut dipoles

$$dN_{\text{cut}} = N l \cos \theta dA = \frac{1}{q} \mathbf{P} \cdot \mathbf{n}, \quad (27)$$

as $P = Nql$.

- b) Suppose the closed surface S is embedded in the dielectric. Then by eq. (27) the net charge bound by the surface

$$\begin{aligned} q_{\text{enc}} &= (-q) \oint_S \frac{dN_{\text{cut}}}{dA} dA \\ &= - \oint_S \mathbf{P} \cdot d\mathbf{A}. \end{aligned} \quad (28)$$

Note that since \mathbf{n} is the outward surface normal, then when $\mathbf{P} \cdot \mathbf{n} > 0$ a negative charge is enclosed, whence the sign above.

c) By Gauss' Law on S

$$\begin{aligned} \oint_S \mathbf{E} \cdot d\mathbf{A} &= 4\pi(q_{\text{free}} + q_{\text{enc}}) \\ &= 4\pi q_{\text{free}} - \oint_S 4\pi \mathbf{P} \cdot d\mathbf{A} , \\ \Rightarrow \oint \mathbf{D} \cdot d\mathbf{A} &= 4\pi q_{\text{free}} , \end{aligned} \quad (29)$$

where $\mathbf{D} = \mathbf{E} + 4\pi \mathbf{P}$.

d) Consider now a magnetized material, whose magnetization we coarse-grain into cells. Similarly to the dielectric, each cell is an ensemble of circular ring current magnetic dipoles, current I and area S , aligned with \mathbf{M} . Suppose we embed a curve in this material, Once again the curve line element is locally flat - i.e. straight - within any cell.

Let us now consider the number of magnetic dipoles threaded by the curve line element $d\mathbf{l}$; i.e. the number the line element passes through. Any dipole whose center is within $\sqrt{S} \cos \theta / \pi$ of the line element is threaded by it, so the line element threads a cylindrical volume $S \cos \theta dl$. Hence the number of threaded dipoles

$$dN_t = NS \cos \theta = \frac{c}{I} \mathbf{M} \cdot d\mathbf{l} , \quad (30)$$

since $M = NIS/c$.

Now, for a closed loop Γ embedded in this magnet, the net current passing through the loop is simply the sum of currents threaded by the loop. A current passes through the loop in a positive sense if it has a right-handed orientation with respect to the loop. Hence the net current passing through the loop

$$I_{\text{bound}} = \oint_{\Gamma} I \frac{dN_t}{dl} dl = c \oint_{\Gamma} \mathbf{M} \cdot d\mathbf{l} . \quad (31)$$

Hence Ampère's law for the loop becomes

$$\begin{aligned} \oint_{\Gamma} \mathbf{B} \cdot d\mathbf{l} &= \frac{4\pi}{c} (I_{\text{free}} + I_{\text{bound}}) \\ &= \frac{4\pi}{c} I_{\text{free}} + \oint_{\Gamma} 4\pi \mathbf{M} \cdot d\mathbf{l} , \\ \Rightarrow \oint \mathbf{H} \cdot d\mathbf{l} &= \frac{4\pi}{c} I_{\text{free}} , \end{aligned} \quad (32)$$

where $\mathbf{H} = \mathbf{B} - 4\pi \mathbf{M}$.

Ex 1.6: Disk Electret

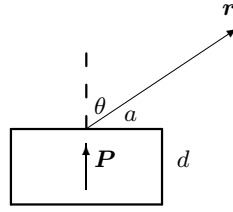
Consider a disk electret as shown in the figure.

Since the polarization is uniform, there is no bound volume charge ρ_b . However, on the top and bottom surface we have bound surface charge

$$\sigma_b = \mathbf{n} \cdot \mathbf{P} = \pm P . \quad (33)$$

Consider the potential at \mathbf{r} , where $r \gg a, d$ (NB: this limit should have been specified in the question), due to the top surface bound charge. By Coloumb, we have potential

$$\begin{aligned} \Phi_1(\mathbf{r}) &= \int_{\text{top}} dr' d\theta' \frac{r' P}{|\mathbf{r} - \mathbf{r}'|} \\ &\simeq \frac{P}{r} \int_{\text{top}} dr' d\theta' = \frac{\pi a^2 P}{r} , \end{aligned} \quad (34)$$



to leading order in a/r . The potential due to the lower disk is similarly

$$\begin{aligned}
 \Phi_2(\mathbf{r}) &\simeq -\frac{\pi a^2 P}{|\mathbf{r} + d\mathbf{e}_z|} \\
 &= -\frac{\pi a^2 P}{\sqrt{r^2 + d^2 + 2rd \cos \theta}} \\
 &\simeq -\frac{\pi a^2 P}{r} \left(1 - \frac{d}{r} \cos \theta\right), \tag{35}
 \end{aligned}$$

expanding in d/r . Hence the potential at \mathbf{r} ,

$$\Phi(\mathbf{r}) = \frac{\pi a^2 P}{r} - \frac{\pi a^2 P}{r} \left(1 - \frac{d}{r} \cos \theta\right) = \frac{\pi a^2 P d \cos \theta}{r^2}. \tag{36}$$

We may rewrite this as $\Phi = Pd\mathbf{A} \cdot \mathbf{e}_r/r^2$, where \mathbf{A} is the area vector of the top surface. However, note that $\mathbf{A} \cdot \mathbf{e}_r = r^2\Omega$, where Ω is the solid angle subtended by \mathbf{A} at \mathbf{r} . Hence

$$\Phi(\mathbf{r}) = Pd\Omega. \tag{37}$$

Finally, if we ignore edge effects then charge distribution on the electret is simply that of a parallel plate capacitor, with charge density $\sigma = P$ on the top plate. The electric field, as in Question 1.1, is simply $\mathbf{E} = -4\pi\mathbf{P}$. Hence within the electret $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \mathbf{0}$.