Homework 4

Laplace Equation (2)

Ex. 4.1: Conducting sphere with net charge

Calculate the potential at all points in space exterior to a conducting sphere of radius a placed in a uniform electric field $\mathbf{E}_0$. The sphere has net charge q.

Ex. 4.2: Spherical surface with prescribed potential

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Ex. 4.3: Depolarizing Factor

Consider an infinitely long dielectric cylinder of radius a and dielectric constant ϵ. The cylinder is placed in an electric field $\mathbf{E}_0$ directed perpendicular to the cylinder's axis.

a) Find the electrostatic potential Φ at all points outside and inside the cylinder.

b) Find the corresponding electric field $\mathbf{E}$.

c) Show that the field inside the cylinder can be written as a superposition of the applied field and the incremental field produced by the polarization induced on the electric cylinder,

$$\mathbf{E} = \mathbf{E}_0 - 4\pi L\mathbf{P},$$

where $\mathbf{P}$ is the polarization in the cylinder and L is the "depolarizing factor". What value does L take?
Note: The difference between the actual field $\mathbf{E}$ inside the cylinder and the applied field $\mathbf{E}_0$ is known as the "depolarizing field".

Ex. 4.4: Demagnetizing field

For an arbitrary stationary current density $\mathbf{j}(\mathbf{r})$, the vector potential $\mathbf{A}(\mathbf{r})$ reads

$$\mathbf{A}(\mathbf{r}) = \int_V dv' \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \tag{1}$$

where the integral is over the volume V in which the current density $\mathbf{j}$ is nonzero. (It would be more precise to state that Eq. (1) represents a possible solution of the vector potential, since there are other realizations of the vector potential that correspond to the same magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$.)

a) Argue that, away from the currents that give rise to the magnetic field, the vector potential $\mathbf{A}$ corresponding to Eq. (1) above satisfies the Laplace equation

$$\Delta \mathbf{A} = 0.$$

b) Consider an infinite cylinder of radius a, made of a magnetic material with relative magnetic permeability μ. The cylinder is placed in a magnetic field $\mathbf{B}_0$ directed perpendicular to the cylinder's axis. Find the vector potential $\mathbf{A}$ and the magnetic field $\mathbf{B}$ at all points inside and outside the cylinder. Show that, for a strongly magnetic material with $\mu \gg 1$ the internal field is essentially twice the applied field $\mathbf{B}_0$.
Hint: Use your answer to Ex. 4.1 a.

c) Now consider the force (per unit length) on the cylinder if the cylinder carries a current I. According to the Lorentz force formula, the force (per unit length $d\mathbf{l}$) on the cylinder is

$$d\mathbf{F} = \frac{I}{c} d\mathbf{l} \times \mathbf{B}.$$

Which magnetic field do we have to take for $\mathbf{B}$, the internal field you calculated in (b) or the external field $\mathbf{B}_0$? Explain your answer.

Ex. 4.5: Conducting Cylinder

Consider a hollow conducting cylinder of radius a and length L, of which the two ends are closed by conducting plates insulated electrically from the walls of the cylinder.

a) Calculate the potential Φ inside the cylinder if the walls of the cylinder and one of the endplates are held at potential $\Phi = 0$, whereas the other endplate is held at potential $-\Phi_0$.

b) Calculate the potential Φ inside the cylinder if the walls of the cylinder are held at potential $\Phi = 0$, whereas both endplates are held at potential $-\Phi_0$.