Solutions: Homework 7

Ex. 7.1: Frustrated Total Internal Reflection

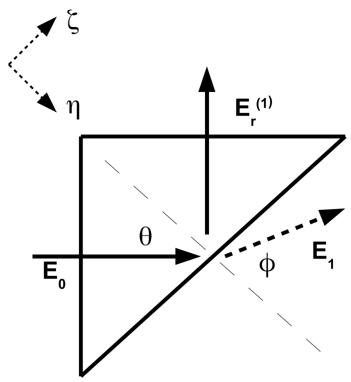
(a) Consider light propagating from a prism, with refraction index n, into air, with refraction index 1. We fix the angle of incidence $\theta = \pi/4$. By Snell's law we have

$$\sin\phi = n\sin\theta = n/\sqrt{2}\ , \tag{1}$$

where ϕ is the angle of refraction. Total internal reflection occurs for $\sin\phi > 1$, so

$$n > \sqrt{2}\ . \tag{2}$$

(b) Consider the electric fields produced at the first interface, as shown in the figure, with coordinates (ζ, η) chosen to be tangential and normal to the refraction surface.



First, we may write the refracted wave as

$$\boldsymbol{E}_1(\boldsymbol{x}, t) = \boldsymbol{E}_1 \exp\left[i\big(k_1\eta\cos\phi + k_1\zeta\sin\phi - \omega t\big)\right] \tag{3}$$

but $\cos\phi = i\sqrt{n^2/2 - 1} \equiv iQ$, since $n > \sqrt{2}$, so we then have

$$\boldsymbol{E}_1(\boldsymbol{x}, t) = \boldsymbol{E}_1 e^{-(\omega/c)Q\eta} \exp\left[i\big(k_1 n\zeta/\sqrt{2} - \omega t\big)\right]\ , \tag{4}$$

which is called an evanescent wave, since one component of the wavevector $\boldsymbol{k}_1$ is now imaginary. Note $k_1 = \omega/c$ in air.

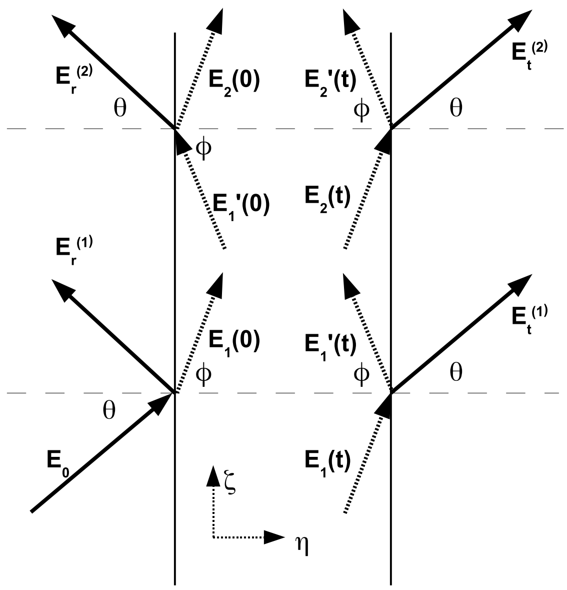
The Fresnel equations, derived for real wavevectors, still hold in the case that the wavevectors (or equivalently the angles of refraction and reflection) become complex. Assuming that the light is polarized perpendicularly to the plane of incidence, we then have

$$E_r^{(1)} = E_0\left(\frac{\cos\theta - (1/n)\cos\phi}{\cos\theta + (1/n)\cos\phi}\right) = E_0\left(\frac{n - i\sqrt{2}Q}{n + i\sqrt{2}Q}\right)\ , \tag{5}$$

and

$$E_1 = E_0\left(\frac{2\cos\theta}{\cos\theta + (1/n)\cos\phi}\right) = E_0\left(\frac{2n}{n + i\sqrt{2}Q}\right)\ . \tag{6}$$

Let the first refraction interface be located at $\eta = 0$, and consider now a second interface at $\eta = t$. At each interface there are reflections and refractions, producing an infinite sequence of such between the



two prisms. The first two sets of reflection-refractions are shown in the figure. Note that by Snell's law, the angle of the transmitted wave is also θ.

In order to calculate the reflected and transmitted amplitudes, in principle we must add all contributions from each set of reflection-refractions. We do this by repeatedly applying the Fresnel equations, and adding a factor of $e^{-(\omega/c)Qt}$ each time we travel from one interface to the other. Hence, writing $E_1 = E_1(0)$, then

$$E_1(t) = E_1(0)e^{-(\omega/c)Qt} \tag{7}$$

and by the Fresnel equations,

$$\begin{aligned} E_t^{(1)} &= E_1(t)\left(\frac{2\cos\phi}{\cos\phi + n\cos\theta}\right) \\ &= E_1(t)\left(\frac{2\sqrt{2}iQ}{n+i\sqrt{2}Q}\right) \\ &= E_0 e^{-(\omega/c)Qt}\left(\frac{2n}{n+i\sqrt{2}Q}\right)\left(\frac{2\sqrt{2}iQ}{n+i\sqrt{2}Q}\right) , \end{aligned} \tag{8}$$

and similarly

$$\begin{aligned} E_1'(t) &= E_1(t)\left(\frac{\cos\phi - n\cos\theta}{\cos\phi + n\cos\theta}\right) \\ &= -E_1(t)\left(\frac{n-i\sqrt{2}Q}{n+i\sqrt{2}Q}\right) \\ &= -E_0 e^{-(\omega/c)Qt}\left(\frac{2n}{n+i\sqrt{2}Q}\right)\left(\frac{n-i\sqrt{2}Q}{n+i\sqrt{2}Q}\right) . \end{aligned} \tag{9}$$

It follows from this that each reflection inside the gap between the prisms adds a factor

$$\rho = -\left(\frac{n-i\sqrt{2}Q}{n+i\sqrt{2}Q}\right) . \tag{10}$$

and transmission back into a prism adds a factor

$$\tau = \left(\frac{2\sqrt{2}iQ}{n + i\sqrt{2}Q}\right) . \tag{11}$$

The other factor

$$\kappa = \left(\frac{2n}{n + i\sqrt{2}Q}\right) \tag{12}$$

arises precisely once, due to the original transmission into the air gap of the evanescent wave. In general, we may then write down the kth order reflected and transmitted waves as

$$\begin{aligned} E_r^{(1)} &= -E_0\rho \\ E_r^{(k)} &= E_0 e^{-(2k-2)(\omega/c)Qt}\rho^{2k-3}\kappa\tau \\ E_t^{(k)} &= E_0 e^{-(2k-1)(\omega/c)Qt}\rho^{2k-2}\kappa\tau \ , \end{aligned} \tag{13}$$

so we have two complex geometric series. Since $|\rho| = 1$, then these series are convergent, and so

$$\begin{aligned} E_r &= -\rho E_0\left(1 - \kappa\tau\frac{e^{-2(\omega/c)Qt}}{1 - \rho^2 e^{-2(\omega/c)Qt}}\right) \\ E_t &= E_0\kappa\tau\frac{e^{-(\omega/c)Qt}}{1 - \rho^2 e^{-2(\omega/c)Qt}} \end{aligned} \tag{14}$$

are the complex amplitudes of the reflected and transmitted waves.

(c) Since E_0, E_r and E_t all exist in the same medium, then the reflection and transmission coefficients are simply

$$\begin{aligned} R &= \left|\frac{E_r}{E_0}\right|^2 \\ T &= \left|\frac{E_t}{E_0}\right|^2 . \end{aligned} \tag{15}$$

The algebra to determine R and T from eqs (14) can be done by Mathematica. One finds

$$\begin{aligned} R &= \frac{(n^2-1)^2\big(1 - e^{-2(\omega/c)Qt}\big)^2}{\big(1 - e^{-2(\omega/c)Qt}\big)^2 + n^2(n^2-2)\big(1 + e^{-2(\omega/c)Qt}\big)^2} \\ T &= \frac{4(n^2-2)^2 e^{-2(\omega/c)Qt}}{\big(1 - e^{-2(\omega/c)Qt}\big)^2 + n^2(n^2-2)\big(1 + e^{-2(\omega/c)Qt}\big)^2} \end{aligned} \tag{16}$$

Note $R + T = 1$, as expected.

For $t \to 0$, we have $R = 0$ and $T = 1$. This is precisely what is expected, since in the limit $t \to 0$ the air gap and the interfaces cease to exist, and so we expect the wave to propagate only forwards. Alternatively, in the limit $t \to \infty$ we have $R = 1$ and $T = 0$, which is total internal reflection. This total internal reflection is expected, since the limit $t \to \infty$ corresponds to the lone prism of part (a).

Alternative: For algebraic simplicity we can also consider eqs (14) in the limit $n \gg \sqrt{2}$, in which $Q \simeq n/\sqrt{2}$. Then in this limit

$$\kappa\tau = \frac{4i}{(1+i)^2} = 2 \ , \tag{17}$$

and

$$\rho = -\frac{1-i}{1+i} = -i \ . \tag{18}$$

It follows immediately from (14) that

$$|E_r| = E_0\left(1 - 2\frac{e^{-2(\omega/c)Qt}}{1+e^{-2(\omega/c)Qt}}\right) = E_0\left(\frac{1-e^{-2(\omega/c)Qt}}{1+e^{-2(\omega/c)Qt}}\right) , \tag{19}$$

$$|E_t| = 2E_0\left(\frac{e^{-(\omega/c)Qt}}{1+e^{-2(\omega/c)Qt}}\right) . \tag{20}$$

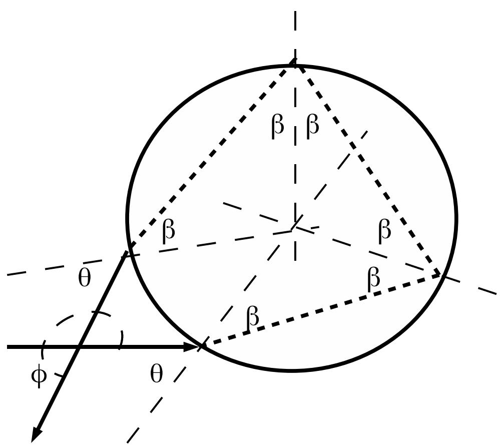
Hence we have reflection and transmission coefficients, for $n \gg \sqrt{2}$,

$$R = \left|\frac{E_r}{E_0}\right|^2 = \left(\frac{1-e^{-2(\omega/c)Qt}}{1+e^{-2(\omega/c)Qt}}\right)^2$$

$$T = \left|\frac{E_t}{E_0}\right|^2 = \frac{4e^{-2(\omega/c)Qt}}{(1+e^{-2(\omega/c)Qt})^2} . \tag{21}$$

Ex. 7.2: Secondary Rainbow

Consider a light ray in air ($n = 1$) incident upon a spherical raindrop, such that it is internally reflected twice, as shown in the figure. Let the refractive index of the water be n, the angle of incidence θ and the angle of refraction β.



Defining the total angle ϕ through which the light ray is refracted in the usual positive sense - i.e. counter-clockwise - then we have

$$\begin{aligned}\phi &= (\theta - \beta) + 2(\pi - 2\beta) + (\theta - \beta) \\ &= 2\theta - 6\beta + 2\pi \\ &= 2\theta - 6\sin^{-1}\left(\frac{1}{n}\sin\theta\right) + 2\pi ,\end{aligned} \tag{22}$$

where the expression for β follows from Snell's law. Note that we have defined the angle ϕ in the opposite sense to the angle of rotation for the primary rainbow.

As we saw for the primary rainbow, the dominant emission of light occurs at $\phi_{\rm min}$. Now,

$$\frac{d\phi}{d\theta} = 2 - \frac{(6/n)\cos\theta}{\sqrt{1-(1/n^2)\sin^2\theta}} \tag{23}$$

and setting $x = \cos\theta$, then the minimum occurs where

$$2 - \frac{6x}{\sqrt{n^2-(1-x^2)}} = 0 . \tag{24}$$

This is easily solved by squaring both sides, so that

$$x^2 = \frac{n^2 - 1}{8} \ , \tag{25}$$

and hence

$$\sin\theta = \sqrt{1 - x^2} = \sqrt{\frac{9 - n^2}{8}} \ . \tag{26}$$

It follows that the minimum angle through which the light ray is rotated is then

$$\phi_{\text{min}} = 2\cos^{-1}\left(\sqrt{\frac{n^2 - 1}{8}}\right) - 6\sin^{-1}\left(\frac{1}{n}\sqrt{\frac{9 - n^2}{8}}\right) + 2\pi \ . \tag{27}$$

Note that by definition, θ and β lie in $[0, \pi/2]$ so we may use $\sin^{-1}$ and $\cos^{-1}$ with the usual branch cuts. For water, $n = 1.33$, one then finds

$$\phi_{\text{min}} = 230^\circ \ . \tag{28}$$

If we define ϕ'_{min} to be the clockwise conjugate angle to ϕ_{min}, in order to match our definition for the primary rainbow in class, then

$$\phi'_{\text{min}} \equiv 2\pi - \phi_{\text{min}} = 130^\circ \ . \tag{29}$$

Now for water, the refractive index $n = n(\omega)$, where ω is the frequency of the incident radiation, and $dn/d\omega > 0$. By the chain rule

$$\frac{d\phi_{\text{min}}}{d\omega} = \frac{d\phi_{\text{min}}}{dn}\frac{dn}{d\omega} \ , \tag{30}$$

and from (27) one can check that $d\phi_{\text{min}}/dn > 0$, so $d\phi_{\text{min}}/d\omega > 0$. Hence for larger ω, ϕ_{min} is larger, and because ϕ is defined in a counterclockwise sense, then blue light will appear at the top of the rainbow, and red at the bottom. This is the opposite orientation to that of the primary rainbow, since ϕ_{min} was defined in a clockwise sense for that case.

Ex. 7.3: Brewster's Angle

Consider light propagating in a material $n = n_1$ incident at angle θ_1 upon the front surface of a dielectric slab with $n = n_2$. Let the angle of refraction be θ_2. Clearly θ_2 is the angle of incidence of the light ray on the rear surface.

If incidence on the front face occurs at the Brewster angle, then

$$\tan\theta_1 = \frac{n_2}{n_1} \implies \sin\theta_1 = \frac{n_2}{\sqrt{n_1^2 + n_2^2}} \ . \tag{31}$$

By Snell's law, the sine of the refraction angle θ_2

$$\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1 = \frac{n_1}{\sqrt{n_1^2 + n_2^2}} \ , \tag{32}$$

so it follows that $\tan\theta_2 = n_1/n_2$. Hence incidence at the rear surface also occurs at the Brewster angle.