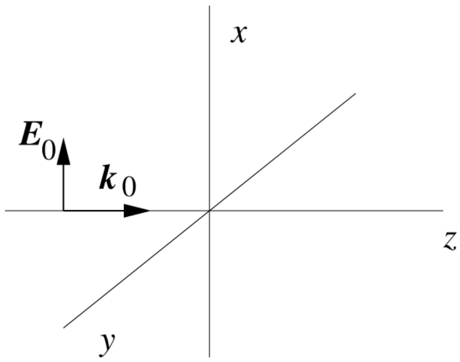
Homework 8

Wave Guides

Ex 8.1: Electromagnetic wave incident on conductor

An electromagnetic wave of frequency ω, moving in a medium with dielectric constant ϵ and magnetic permeability $\mu = 1$, is incident normally on the surface of a metal. The metal has finite but large conductivity σ_m and dielectric constant ϵ_m.



We choose coordinate axes such that the surface of the metal is the xy plane and the metal fills the half-space $z > 0$, see figure. The electric field of the incident wave is $\mathbf{E}_0(\mathbf{r}, t) = \mathrm{Re} E_0^0 e^{i(kz-\omega t)} \mathbf{e}_x$, with $z < 0$. The frequency ω satisfies the inequality $\omega \ll \sigma_m$, and you may give all your answers below to leading order in the small ratio ω/σ_m. In the questions below, indicate directions using a right-handed coordinate system with unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_x$.

a) What are the electric and magnetic fields $\mathbf{E}_1(\mathbf{r}, t)$ and $\mathbf{B}_1(\mathbf{r}, t)$ of the reflected wave? Write your answer in terms of the amplitude E_0, wavenumber k, and frequency ω of the incident wave and the material properties $(\epsilon, \epsilon_m, \sigma_m)$.

b) The electric field of the transmitted (refracted) wave reads

$$\mathbf{E}(\mathbf{r}, t) = \mathrm{Re} E_2^0 e^{i(k_m z-\omega t)} \mathbf{e}_x,$$

where k_m is a complex wave number. Find expressions for the complex field amplitude E_2^0 and the wave number k_m in terms of the amplitude E_0 and frequency ω of the incident wave and the material properties $(\epsilon, \epsilon_m, \sigma_m)$. It is sufficient to give your answer to leading order in ω/σ_m.

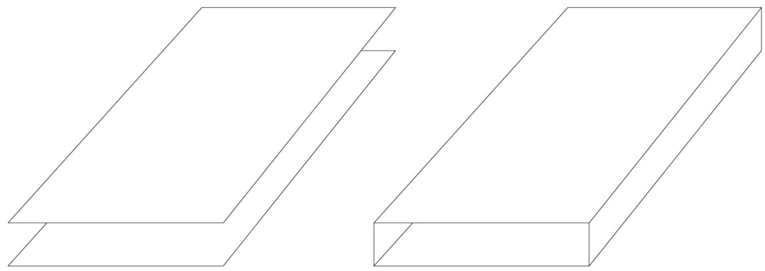
c) What is the magnetic field $\mathbf{B}(\mathbf{r}, t)$ of the transmitted wave? You may express your answers in terms of E_2^0 and k_m.

d) Find an expression for the current density $j(\mathbf{r}, t)$ in the metal. You may express your answer in terms of E_2^0 and k_m.

e) What is the time-averaged Poynting vector for the transmitted wave? How do you write the energy conservation law in this case?

Ex. 8.2: Parallel Conducting Planes

In Sex. 7.2 of Heald & Marion you can find a discussion how TE waves can propagate between two perfectly conducting plates (figure below, left). But TE waves are not the only electromagnetic waves that can propagate between two conducting planes. Show that two unbounded conducting plates also support TEM waves. How can you reconcile this observation with the fact that a rectangular waveguide consisting of two conducting planes of a finite width and narrow conducting strips at each end does not support TEM waves (figure below, right)?

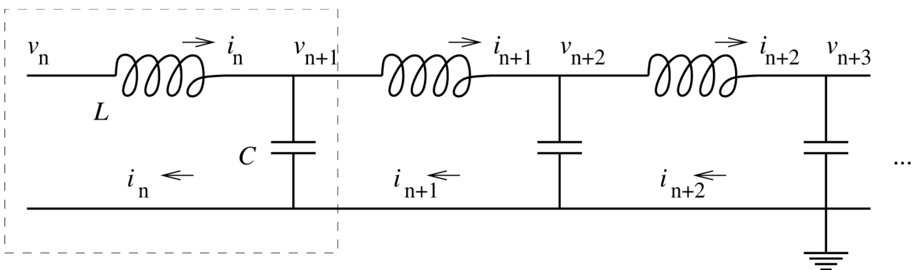


Ex. 8.3: Electric and Magnetic Fields for a Transmission Line

Heald & Marion Ex 7-5

Ex. 8.4: Discrete element representation of a transmission line

One can represent a transmission line with a capacitance C_l per unit length and an inductance L_l per unit length as a discrete ladder network of conductors and capacitors. An example of such a representation is given in the figure below. In the figure, each segment of the ladder (as indicated by the dashed box) has length δz, and consists of a capacitor of capacitance $C = C_l \Delta z$ and an inductor of inductance $L = L_l \Delta z$.



a) The figure indicates voltages v_n at the nodes of the network and currents i_n at the links of the network. (Note that the lower conductor is grounded.) Show that these satisfy the recursion relation

$$v_{n+1} - v_n = i\omega L i_n, \qquad i_{n+1} - i_n = i\omega C v_{n+1}.$$

b) Show that these recursion relations simplify to the differential equations for $v(z)$ and $i(z)$ for a continuous transmission line that are discussed in Sec 7.1 of Heald & Marion

c) Calculate the impedance Z of an infinite ladder, and show that $Z \to \sqrt{L_l/C_l}$ in the limit of a continuous transmission line. *Hint: In order to find Z, argue that the impedance Z of an infinite ladder does not change upon addition of a single segment, and then derive a self-consistent equation for Z.*

d) You can model a transmission line with losses by adding a resistance $R = R_l \Delta z$ in series with each inductance L. How does that affect the recursion relations you derived in (a)? What is the solution of these equations in the continuum limit?

Suggested Heald & Marion Problems for further study:

7-1, 7-3, 7-11