Solutions: Homework 8

Ex. 8.1: EM Waves Incident on a Conductor

Consider a conductor lying in the $z > 0$ half space with conductivity σ_m and dielectric constant ϵ_m, and a dielectric in the $z > 0$ half space with dielectric constant ϵ. A wave $\boldsymbol{E}_0(\boldsymbol{r}, t) = E_0^0 e^{i(kz-\omega t)}\boldsymbol{e}_x$ is normally incident on the conductor, and $\omega \ll \sigma_m$. We also assume $\epsilon_m \lesssim 4\pi$.

(a) First, all the results for normal incidence hold in this case, but with the refractive index of the conductor now a complex number. The refractive index of the dielectric

$$n_1 = \sqrt{\epsilon}\ , \tag{1}$$

and complex refractive index of the conductor

$$\begin{aligned} n_m &= \sqrt{\epsilon_m + i\frac{4\pi\sigma_m}{\omega}} \\ &\simeq (1+i)\sqrt{\frac{2\pi\sigma_m}{\omega}} \\ &\equiv (1+i)\alpha^{-1} \end{aligned} \tag{2}$$

to leading order in $\epsilon_m\omega/\sigma_m$. Note here $\alpha \ll 1$.

The complex magnitude of the reflected electric field $\boldsymbol{E}_1(\boldsymbol{r}, t) = E_1^0 e^{i(kz-\omega t)}\boldsymbol{e}_x$ is then

$$\begin{aligned} \frac{E_1^0}{E_0^0} &= \frac{n_m - n_1}{n_m + n_1} \\ &= \frac{1 - \sqrt{\epsilon}\alpha(1-i)/2}{1 + \sqrt{\epsilon}\alpha(1-i)/2} \\ &\simeq \left[1 - \sqrt{\epsilon}\alpha(1-i)/2\right]^2 \\ &\simeq 1 - \sqrt{\epsilon}\alpha(1-i)\ , \end{aligned} \tag{3}$$

to leading order in α. Taking real parts then

$$\boldsymbol{E}_1(\boldsymbol{r}, t) = E_0^0\Big[\big(1 - \sqrt{\epsilon}\alpha\big)\cos[kz-\omega t] - \sqrt{\epsilon}\alpha\sin[kz-\omega t]\Big]\boldsymbol{e}_x\ . \tag{4}$$

Since the reflected field propagates in the $-z$ direction, the magnetic field $\boldsymbol{B}_1 = -n_1\boldsymbol{e}_z \times \boldsymbol{E}_1$, so

$$\boldsymbol{B}_1(\boldsymbol{r}, t) = -\sqrt{\epsilon}E_0^0\Big[\big(1 - \sqrt{\epsilon}\alpha\big)\cos[kz-\omega t] - \sqrt{\epsilon}\alpha\sin[kz-\omega t]\Big]\boldsymbol{e}_y\ . \tag{5}$$

(b) Let the transmitted wave be $\boldsymbol{E}_2(\boldsymbol{r}, t) = E_2^0 e^{i(k_m z-\omega t)}\boldsymbol{e}_x$. Then

$$\begin{aligned} \frac{E_2^0}{E_0^0} &= \frac{2n_1}{n_m + n_1} \\ &= \frac{\sqrt{\epsilon}(1-i)\alpha}{1 + \sqrt{\epsilon}\alpha(1-i)/2} \\ &\simeq \sqrt{\epsilon}(1-i)\alpha = (1-i)\sqrt{\epsilon\omega/2\pi\sigma_m} \end{aligned} \tag{6}$$

to leading order. The complex wavenumber

$$k_m = n_m\frac{\omega}{c} = \frac{\omega}{c}(1+i)\alpha^{-1} = (1+i)\sqrt{2\pi\sigma_m\omega/c^2} \equiv (1+i)\delta^{-1}\ . \tag{7}$$

(c) The magnetic field of the transmitted wave $\boldsymbol{B}_2 = n_m \boldsymbol{e}_z \times \boldsymbol{E}_2$. Hence

$$\boldsymbol{B}_2(\boldsymbol{r}, t) = \frac{k_m c}{\omega} E_2^0 e^{i(k_m z - \omega t)} \boldsymbol{e}_y \ , \tag{8}$$

in terms of only E_2^0 and k_m.

(d) Applying Ohm's Law, the current density in the conductor $\boldsymbol{j} = \sigma_m \boldsymbol{E}_2$. Hence,

$$\boldsymbol{j} = \sigma_m E_2^0 e^{i(k_m z - \omega t)} \boldsymbol{e}_x \ . \tag{9}$$

After taking real parts we have

$$\boldsymbol{j}(\boldsymbol{r}, t) = \sigma_m E_0^0 \sqrt{\epsilon\omega/2\pi\sigma_m} e^{-z/\delta} \Big[\cos(\delta^{-1} z - \omega t) + \sin(\delta^{-1} z - \omega t) \Big] \ . \tag{10}$$

(e) The time-averaged Poynting vector

$$\begin{aligned} \langle \boldsymbol{S} \rangle &= \frac{c}{8\pi} \boldsymbol{E}_2 \times \boldsymbol{B}_2^* \\ &= \frac{c}{8\pi} e^{-2z/\delta} |E_2^0|^2 n_m^* \boldsymbol{e}_z \\ &= \frac{\epsilon}{8\pi^2} \frac{c^2}{\delta\sigma_m} (E_0^0)^2 e^{-2z/\delta} \boldsymbol{e}_z \end{aligned} \tag{11}$$

after taking the real part only. Since there are free currents flowing in the conductor, then applying Ohm's Law $\boldsymbol{j} = \sigma_m \boldsymbol{E}$, the energy conservation law is

$$\frac{\partial}{\partial t} \mathcal{E} + \nabla \cdot \boldsymbol{S} + \sigma_m \boldsymbol{E}^2 = 0 \ . \tag{12}$$

Ex. 8.2: Parallel Conducting Planes

Consider two conducting planes, located at $x = 0$ and $x = d$, so their common surface normal is $\boldsymbol{e}_x$. The boundary condition for the two planes is

$$\boldsymbol{e}_x \times \boldsymbol{E} = 0 \ , \quad \text{and } \boldsymbol{e}_x \cdot \boldsymbol{B} = \boldsymbol{0} \ . \tag{13}$$

Clearly, a TEM mode with $\boldsymbol{k} = k\boldsymbol{e}_z$, $\boldsymbol{E} = E\boldsymbol{e}_x$ and $\boldsymbol{B} = B\boldsymbol{e}_y$ satisfies the boundary conditions, so these plane support TEM modes.

The proof that uniform cross-section hollow conductors do not support TEM modes depends on the conductor's surface being closed (i.e. having no boundary) and connected. However, in the case of the two planes, the conducting surface is clearly not connected, so we do not expect TEM modes to be forbidden.

Ex. 8.3: E & M Fields for a Transmission Line

(a) Consider a coaxial cable with inner radius a and outer radius b, with the inner conductor at a potential V_0 with respect to the outer one. As found in previous homework, in the cylindrical coordinates (r, θ, z) the potential for $a \le r \le b$ is

$$\Phi(r) = -V_0 \frac{\ln(r/a)}{\ln(b/a)} \ , \tag{14}$$

and hence the electric field

$$\boldsymbol{E}(r) = -\nabla\Phi(r) = \frac{V_0}{\ln(b/a)} \frac{\boldsymbol{e}_r}{r} \ . \tag{15}$$

(b) Assuming the the current I_0 flows along the inner conductor in the $+z$ direction, then by Ampère's Law in integral form, clearly the magnetic field for $a \leq r \leq b$ is

$$\boldsymbol{B}(r) = \frac{2I_0}{c}\frac{\boldsymbol{e}_\theta}{r} \ . \tag{16}$$

(c) Let $V(z,t) = V_0 e^{i(kz-\omega t)}$ and $I(z,t) = I_0 e^{i(kz-\omega t)}$, and $Z_0 = V_0/I_0$. Consider the dynamical fields

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{V(z,t)}{\ln(b/a)}\frac{\boldsymbol{e}_r}{r} = \frac{V_0}{\ln(b/a)}\frac{e^{i(kz-\omega t)}}{r}\boldsymbol{e}_r \tag{17}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{2I(z,t)}{c}\frac{\boldsymbol{e}_\theta}{r} = \frac{2I_0}{c}\frac{e^{i(kz-\omega t)}}{r}\boldsymbol{e}_\theta \ . \tag{18}$$

which are obtained by näively replacing $V_0 \to V(z,t)$ and $I_0 \to I(z,t)$. Note these fields have not been derived from Maxwell's equations, so the idea is to check that they satisfy the usual wave equations, and hence are the correct fields generated by $V(z,t)$ and $I(z,t)$.

In the cylindrical coordinates chosen, we may write the electric field $\boldsymbol{E} = E_r(r,z)\boldsymbol{e}_r$. Using the hint provided, the Laplacian

$$\begin{aligned}
\nabla^2\boldsymbol{E} &= \nabla\big(\nabla\cdot\boldsymbol{E}\big) - \nabla\times\big(\nabla\times\boldsymbol{E}\big) \\
&= \boldsymbol{0} - \nabla\times\left(\frac{\partial}{\partial z}E_r(r,z)\right)\boldsymbol{e}_\theta \\
&= \left(ik\frac{\partial}{\partial z}\big[rE_r(r,z)\big]\right)\frac{\boldsymbol{e}_r}{r} \\
&= -k^2\boldsymbol{E} \ ,
\end{aligned} \tag{19}$$

noting that $\partial/\partial r(rE_r) = 0$. Hence

$$\nabla^2\boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{E} = -k^2\boldsymbol{E} + \frac{\omega^2}{c^2}\boldsymbol{E} = \boldsymbol{0} \ , \tag{20}$$

provided $k = \omega/c$, and so the electric field (17) satisfies the wave equation.

Similarly writing the magnetic field as $\boldsymbol{B} = B_\theta(r,z)\boldsymbol{e}_\theta$, then

$$\begin{aligned}
\nabla^2\boldsymbol{B} &= \nabla\big(\nabla\cdot\boldsymbol{B}\big) - \nabla\times\big(\nabla\times\boldsymbol{B}\big) \\
&= \boldsymbol{0} + \nabla\times\left(\frac{\partial}{\partial z}B_\theta(r,z)\right)\boldsymbol{e}_r \\
&= \left(ik\frac{\partial}{\partial z}B_\theta(r,z)\right)\boldsymbol{e}_\theta \\
&= -k^2\boldsymbol{B} \ ,
\end{aligned} \tag{21}$$

noting that $\partial/\partial r(rB_\theta) = 0$. Hence

$$\nabla^2\boldsymbol{B} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{B} = -k^2\boldsymbol{B} + \frac{\omega^2}{c^2}\boldsymbol{B} = \boldsymbol{0} \ , \tag{22}$$

provided $k = \omega/c$, and so the magnetic field (18) satisfies the wave equation.

(d) The time-averaged Poynting vector

$$\begin{aligned}
\langle\boldsymbol{S}\rangle &= \frac{c}{8\pi}\boldsymbol{E}\times\boldsymbol{B}^* \\
&= \frac{1}{4\pi}\frac{V_0 I_0}{\ln(b/a)}\frac{\boldsymbol{e}_z}{r^2} \ .
\end{aligned} \tag{23}$$

Hence the power transmitted through the interior cross section of the cable by the EM field is

$$\begin{aligned} P &= \int \boldsymbol{S} \cdot d\boldsymbol{A} \\ &= \frac{1}{4\pi} \frac{V_0 I_0}{\ln(b/a)} \int_0^{2\pi} \int_a^b \frac{dr}{r} \\ &= \frac{V_0 I_0}{2} \ . \end{aligned} \tag{24}$$

Since the electric and magnetic fields outside the cable are zero, then P is the total power transmitted down the line by the EM wave. Note that P is the same as the rms power dissipated by the impedance Z_0, which is $I_0^2 Z_0/2$.

Ex. 8.4: Discrete Element Representation of a Transmission Line

(a) Let the nth inductor be L_n. The current running through L_n is I_n, and the impedance $Z_L = -i\omega L$, assuming the ladder is driven by an external sinusoidal voltage or current source $\sim e^{-i\omega t}$. *Note that the $-$ sign in the exponential means that Z_L is the opposite sign to the convention you may have seen elsewhere!* From the defined direction of I_n, the voltage drop over L_n is $V_n - V_{n+1}$, so by Ohm's Law ($V = IZ$) we have

$$V_{n+1} - V_n = i\omega L I_n \ . \tag{25}$$

Similarly, let the nth capacitor be C_n, which has impedance $Z_C = i/(\omega C)$ (different again by a sign to the usual convention). Clearly the voltage drop over C_n is by definition V_{n+1}, and by Kirchoff's junction rule, the current flowing through C_n over the voltage drop is $I_n - I_{n+1}$. Hence by Ohms' Law

$$I_{n+1} - I_n = i\omega C V_{n+1} \ . \tag{26}$$

(b) First, note that to discretize the ladder we define the location of the nth node of the ladder to be z_n, and let $z_{n+1} = z_n + \Delta z$. Letting $V_n = V(z_n)$ $[I_n = I(z_n)]$ be the voltage at [current into] the n node, then we have the Taylor expansions

$$V_{n+1} = V_n + \frac{\partial V}{\partial z}\Delta z + \mathcal{O}(\Delta z^2) \ , \tag{27}$$

$$I_{n+1} = I_n + \frac{\partial I}{\partial z}\Delta z + \mathcal{O}(\Delta z^2) \ . \tag{28}$$

Let the capacitance (inductance) per unit length be C_l (L_l), so $C = C_l \Delta z$ and $L = L_l \Delta z$. The recursion relation (25) then becomes

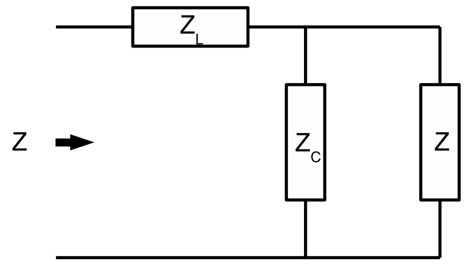
$$\frac{V_{n+1} - V_n}{\Delta z} = i\omega I_n L_l = -\frac{\partial I}{\partial t}(z_n) L_l \ . \tag{29}$$

since $I \sim e^{-i\omega t}$. But in the limit of a continuous transmission line, $\Delta z \to 0$, clearly from (27) $(V_{n+1} - V_n)/\Delta z \to \partial V/\partial z$. Hence

$$\frac{\partial V}{\partial z} = -\frac{\partial I}{\partial t} L_l \ . \tag{30}$$

Similarly, we find from (26) and (28) that

$$\frac{\partial I}{\partial z} = -\frac{\partial V}{\partial t} C_l \ . \tag{31}$$



(c) The impedance of the ladder may be expressed as a repeating continued fraction, so adding an extra ladder element to the infinite ladder does not change its limiting impedance, assuming that such limit exists. Hence, consider the impedance of the circuit shown, in which an extra ladder element is added to the ladder.

Since Z is the impendance of an infinite ladder, then it is unchanged by the addition of the element, so it follows that

$$Z = Z_L + \frac{Z_C Z}{Z_C + Z} \ , \tag{32}$$

as Z_L is in series with the parallel combination of Z_C and Z. Hence Z satisfies

$$Z^2 - Z_L Z - Z_L Z_C = 0 \tag{33}$$

which has solution

$$Z = \frac{1}{2}\left(Z_L \pm \sqrt{Z_L^2 + 4Z_L Z_C}\right) \ . \tag{34}$$

Now, $Z_L = -i\omega L_l \Delta z$, and $Z_C = i(\omega C_l \Delta z)$, so in the limit $\Delta z \to 0$ only the $Z_L Z_C$ term survives. Hence for the continuum limit $\Delta z \to 0$,

$$Z \to \sqrt{L_l/C_l} \ . \tag{35}$$

(d) If a resistance $R = R_l \Delta z$ is added in series with the inductor, the the impedance $Z_L = -i\omega L + R$. The recursion relation (25) then becomes

$$V_{n+1} - V_n = (i\omega L - R) I_n \ , \tag{36}$$

and (26) remains the same.

In the continuum limit, the differential equations are then modified to be

$$\frac{\partial V}{\partial z} = -\frac{\partial I}{\partial t}\big(L_l + iR_l/\omega\big) \ , \frac{\partial I}{\partial z} = -\frac{\partial V}{\partial t} C_l \ , \tag{37}$$

which may be rewritten as

$$\begin{aligned} \frac{\partial^2 V}{\partial z^2} - \big(L_l + iR_l/\omega\big) C_l \frac{\partial^2 V}{\partial t^2} &= 0 \ , \\ \frac{\partial^2 I}{\partial z^2} - \big(L_l + iR_l/\omega\big) C_l \frac{\partial^2 I}{\partial t^2} &= 0 \end{aligned} \tag{38}$$

The solution to these are $V(z,t) = V_0 e^{i(kz-\omega t)}$ and $I(z,t) = I_0 e^{i(kz-\omega t)}$, with dispersion relation

$$k = \sqrt{C_l\big(\omega^2 L_l + i\omega R\big)} \ . \tag{39}$$

Note that k is complex, so that I and V are exponentially suppressed - or damped - in the z direction, as expected.