Solutions: Homework 9

Ex. 9.1: TM Modes in a Rectangular Waveguide

(a) Components of the electric field tangential to the conducting surface must be zero, so $E_z(x, y)$ must satisfy

$$E_z(0, y) = E_z(a, y) = E_z(x, 0) = E_z(x, b) = 0 \ . \tag{1}$$

(b) In general the z component of the electric field has form $E_z(\boldsymbol{x}, t) = E_z(x, y)e^{i(kz-\omega t)}$, so that $E_z(x, y)$ is a solution of the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) E_z = 0 \ , \tag{2}$$

where $k_c^2 = -k^2 + \omega^2/c^2$. Applying the boundary conditions (1), the solution is simply

$$E_z(x, y) = E_z^0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \ , \tag{3}$$

for m, n integers, provided that

$$k_c^2 = \pi^2 \left(m^2/a^2 + n^2/b^2\right) \ . \tag{4}$$

We must have $E_z > 0$ for TM modes, and hence $m, n \in \mathbb{Z}^+$.

(c) The dispersion relation is then

$$k(\omega) = \sqrt{\omega^2/c^2 - k_c^2} = \sqrt{\frac{\omega^2}{c^2} - \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)} \ . \tag{5}$$

The cutoff frequency ω_c is defined by $k(\omega_c) = 0$, so that for the TM_{mn} mode

$$\omega_c = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \ . \tag{6}$$

(d) Applying Maxwell's equations and $B_z = 0$, the transverse electric field components are

$$\begin{aligned} E_x &= \frac{ik}{k_c^2}\frac{\partial}{\partial x}E_z \\ &= \frac{ikE_z^0}{k_c^2}\frac{m\pi}{a}\cos(m\pi x/a)\sin(n\pi y/b) \end{aligned} \tag{7}$$

$$\begin{aligned} E_y &= \frac{ik}{k_c^2}\frac{\partial}{\partial y}E_z \\ &= \frac{ikE_z^0}{k_c^2}\frac{n\pi}{b}\sin(m\pi x/a)\cos(n\pi y/b) \ . \end{aligned} \tag{8}$$

Further, the transverse magnetic field components are

$$\begin{aligned} B_x &= -\frac{i\omega}{ck_c^2}\frac{\partial}{\partial y}E_z \\ &= \frac{i\omega E_z^0}{ck_c^2}\frac{n\pi}{b}\sin(m\pi x/a)\cos(n\pi y/b) \end{aligned} \tag{9}$$

$$\begin{aligned} B_y &= -\frac{i\omega}{ck_c^2}\frac{\partial}{\partial x}E_z \\ &= \frac{i\omega E_z^0}{ck_c^2}\frac{m\pi}{a}\cos(m\pi x/a)\sin(n\pi y/b) \ . \end{aligned} \tag{10}$$

(e) The time-averaged Poynting vector for the TM_{mn} mode

$$\begin{aligned}\langle \boldsymbol{S}\rangle &= \frac{c}{8\pi}\mathrm{Re}\Big(\boldsymbol{E}\times\boldsymbol{B}^*\Big)\\ &= \frac{c}{8\pi}\mathrm{Re}\Big[\Big(E_xB_y^* - E_yB_x^*\Big)\boldsymbol{e}_z + E_zB_x^*\boldsymbol{e}_y - E_zB_y^*\boldsymbol{e}_x\Big]\\ &= \frac{c}{8\pi}\mathrm{Re}\Big[\Big(E_xB_y^* - E_yB_x^*\Big)\boldsymbol{e}_z\Big] \ , \end{aligned} \tag{11}$$

since the other terms are purely imaginary, due to the factors of i in the transverse field components. Hence

$$\begin{aligned}\langle \boldsymbol{S}\rangle = \frac{\omega k E_z^0}{8\pi k_c^2}\Bigg[&\left(\frac{m\pi}{a}\right)^2\cos^2(m\pi x/a)\sin^2(n\pi y/b)\\ &+\left(\frac{n\pi}{b}\right)^2\sin^2(m\pi x/a)\cos^2(n\pi y/b)\Bigg]\boldsymbol{e}_z \ . \end{aligned} \tag{12}$$

Averaging over the cross-section, note

$$\frac{1}{a}\int_0^a \cos^2\left(\frac{m\pi x}{a}\right)dx = \frac{1}{b}\int_0^b \cos^2\left(\frac{n\pi y}{b}\right)dy = \frac{1}{2} \ , \tag{13}$$

so that the cross-section-and-time-averaged Poynting vector is

$$\begin{aligned}\Big\langle\langle \boldsymbol{S}\rangle\Big\rangle &= \frac{1}{ab}\int_A dxdy\langle \boldsymbol{S}\rangle\\ &= \frac{\omega k E_z^0}{32\pi k_c^2}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]\boldsymbol{e}_z\\ &= \frac{\omega k E_z^0}{32\pi}\boldsymbol{e}_z \ . \end{aligned} \tag{14}$$

Ex. 9.2: Waveguide with Circular Cross Section

Consider a waveguide with circular cross section, radius a. For a TE mode, in cylindrical coordinates (r,θ,z) we expect $\boldsymbol{E} = \boldsymbol{E}_0(r,\theta)e^{i(kz-\omega t)}$, and so the wave equation becomes the Helmholtz equation

$$\left(r\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial\theta^2} + k_c^2r^2\right)B_z^0(r,\theta) = 0 \ , \tag{15}$$

where $k_c^2 = \omega^2/c^2 - k^2$, and the boundary condition is

$$\left.\frac{\partial B_z^0}{\partial r}\right|_{r=a} = 0 \ . \tag{16}$$

Applying separation of variables as usual, $B_z^0(r,\theta) = B_z^0(r)e^{\pm in\theta}$ and (15) becomes Bessel's equation

$$\left(r\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + k_c^2r^2 - n^2\right)B_z^0(r,\theta) = 0 \ . \tag{17}$$

As we saw previously, the general solution is $B_z^0(r) = \sum_{m,n}[A_{mn}J_n(k_{c_m}r) + C_{mn}N_n(k_{c_m}r)]$, but since B_z^0 must be regular at $r=0$ then $C_{mn}=0$. The general solution is then

$$B_z^0(r,\theta) = \sum_{m,n} A_{mn}J_n(k_{c_m}r)e^{\pm i(n\theta)} \ , \tag{18}$$

and the TE_{mn} mode, after taking real parts, is

$$B_z^0(r,\theta) = A_{mn}J_n(k_{c_m}r)\cos(n\theta) \ . \tag{19}$$

Note the mode is doubly degenerate, since we may pick the sign of the exponential to be positive or negative, but the real part of B_z^0 is the same in either case. We also have the boundary condition for the TE_{mn} mode

$$\frac{d}{dr}J_n(k_{c_m}a) = 0 \ . \tag{20}$$

By convention $k_{c_{m+1}} > k_{c_m}$, and hence $k_{c_m}a$ is the mth local extremum (maximum or minimum) of J_n.

By definition, the lowest TE mode is that mode which has smallest k_c. It follows that the lowest mode must be a TE_{1n} mode which has the smallest k_{c_1} satisfying (20). Or in other words, the lowest mode corresponds to the Bessel function $J_n(u)$ whose first local extremum occurs at the smallest possible u. The first local extremum of $J_1(u)$ occurs at $u = 1.841$, which is smaller than the first local extrema of any other Bessel-J function. This can easily be seen from a plot of the first few Bessel functions. Hence, the lowest TE mode is the TE_{11} mode ($n = 1$),

$$B_z^0(r,\theta) = B^0 J_1(k_c r)\cos\theta \ , \tag{21}$$

such that $k_c a = 1.841$ and $B^0 = A_{11}$.

Now, to find the transverse field components of TE_{11}, it is convenient to restore the imaginary part of B_z^0. The mode is doubly denegerate, i.e. $B_z^0 = B^0 J_1(k_c r)e^{\pm i\theta}$, so choosing the sign in the exponential to be positive, for the magnetic field components we have

$$\begin{aligned}
\boldsymbol{B}_t &= \frac{k}{ik_c^2}\nabla B_z^0 \\
&= \frac{k}{ik_c^2}\left(\frac{\partial B_z^0}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial B_z^0}{\partial\theta}\boldsymbol{e}_\theta\right) \\
&= \frac{kB^0}{k_c^2}\left[\left(\frac{1}{r}J_1(k_c r) - k_c J_2(k_c r)\right)\sin\theta\boldsymbol{e}_r + \frac{1}{r}J_1(k_c r)\cos\theta\boldsymbol{e}_\theta\right] \\
&\equiv B_r\boldsymbol{e}_r + B_\theta\boldsymbol{e}_\theta \ ,
\end{aligned} \tag{22}$$

where we have taken the real parts. Further, the transverse electric field satisfies $\boldsymbol{e}_z \times \boldsymbol{E}_t = (\omega/ck)\boldsymbol{B}_t$. Writing $\boldsymbol{E}_t = E_r\boldsymbol{e}_r + E_\theta\boldsymbol{e}_\theta$ and taking the cross product with $\boldsymbol{e}_z$, it follows that

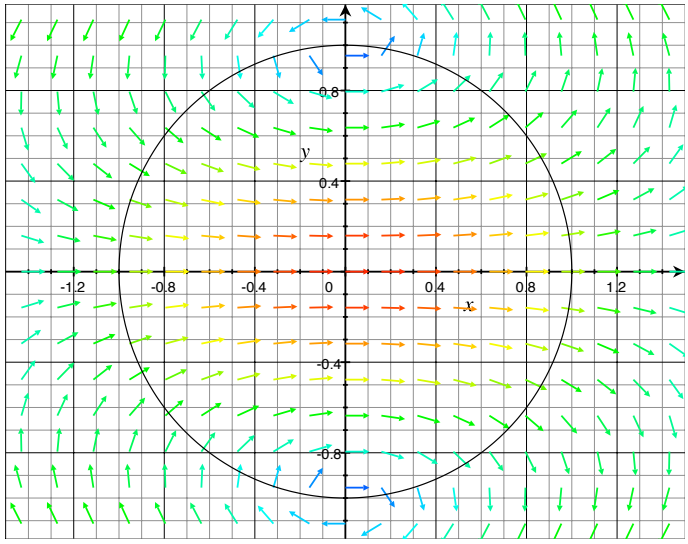
$$E_r\boldsymbol{e}_\theta - E_\theta\boldsymbol{e}_r = \frac{\omega}{ck}\big(B_r\boldsymbol{e}_r + B_\theta\boldsymbol{e}_\theta\big) \ , \tag{23}$$

so $E_r = (\omega/ck)B_\theta$ and $E_\theta = -(\omega/ck)B_r$. That is,

$$\boldsymbol{E}_t = \frac{\omega B^0}{ck_c^2}\left[\frac{1}{r}J_1(k_c r)\cos\theta\boldsymbol{e}_r - \left(\frac{1}{r}J_1(k_c r) - k_c J_2(k_c r)\right)\sin\theta\boldsymbol{e}_\theta\right] \ . \tag{24}$$

Note that the transverse components for the other degenerate TE_{11} mode are the same up to a change of sign for the $\cos\theta$ terms.

A colour plot of the electric field for one of the denegerate TE_{11} modes is below, with red indictating longer vectors, and blue shorter. Note the electric field is normal to the waveguide surface, indicated by the black circle.


0.8
y
0.4
-1.2
-0.8
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x
-0.4
-0.8