\[ \vec{E}_0 \parallel \text{ plane of incidence} \]

\[ R_{\parallel} = \frac{\tan^2(\theta_0 - \theta_0)}{\tan^2(\theta_0 + \theta_0)} \quad T_{\parallel} = 1 - R_{\parallel} \]

Define degree of polarization of reflected wave:

\[ P = \frac{R_\perp - R_{\parallel}}{R_\perp + R_{\parallel}} \quad P = 0 \text{ if } \theta_0 = 0, \frac{\pi}{2} \quad P > 0 \text{ otherwise} \]

\[ P(\theta_0) = 1 \]

---

**total internal reflection**

1) What if \( N_\perp > N_\parallel \)?

Then \( \sin \theta_2 = \frac{N_\perp}{N_\parallel} \sin \theta_0 \) has no solution if \( \theta_0 > \arcsin \left( \frac{N_\parallel}{N_\perp} \right) \)

For \( \theta_0 > \theta_c \) no transmitted wave is possible. Reflection is perfect/total.

2) Let's work this out in greater detail:

\[ \vec{E}_0 = \vec{E}_0 e^{i(k_0x - wt)} \quad \text{if} \quad k_0 \text{ in xy plane } j = 0, 1, 2 \]

\[ \vec{E}_1 = \vec{E}_1 e^{i(k_0x - wt)} \quad \text{we must have} \]

\[ k_{20x} = k_{1x} = k_{2x} \quad \text{what is } k_{2z} ? \]
\[ k_2^2 = \left(\frac{\omega}{c}\right)^2 n_{II}^2 = k_{2x}^2 + k_{2z}^2 = k_{ox}^2 + k_{oz}^2 = k_c^2 \sin^2 \theta_c + k_{2z}^2 = \left(\frac{\omega}{c}\right)^2 n_{II}^2 \sin^2 \theta_c + k_{2z}^2 \]

\[ k_{2z} = \frac{\omega}{c} \sqrt{n_{II}^2 - n_{II}^2 \sin^2 \theta_c} = \frac{\omega}{c} n_{II} \sqrt{1 - \frac{\sin^2 \theta_c}{\sin^2 \theta_c}} \]

\[ \sin \theta_c = \frac{n_{II}}{n_{II}} \]

If \( \theta_c > \theta_c \): \[ k_{2z} = \frac{i \omega}{c} n_{II} \sqrt{\frac{\sin^2 \theta_c}{\sin^2 \theta_c}} - 1 = i k_2 Q \]

\[ \vec{E}_2 = \vec{E}_0 e^{-i k_{ox} x - i \omega t - Q k_2 z} \]

Wave propagation along \( x \) axis (surface) decaying into medium along \( z \) axis.

Phase velocity at \( \theta_c = \theta_c \): \[ \frac{\omega}{k_{ox}} = \frac{c}{n_{II}} \]

At \( \theta_c = \frac{\pi}{2} \): \[ \frac{\omega}{k_{ox}} = \frac{c}{n_{II}} \]

Decay length \[ \frac{1}{\text{Im} k_2 z} = \frac{1}{Q k_2 z} \]

Can show that \[ R = \frac{|E_2|^2}{|E_0|^2} = 1 \]

3) Energy flow in medium II

(Good exercise in Complex Analysis. So good that both Heald & Moring & Jackson don't get it right.)

\[ \langle \mathcal{S} \rangle = \frac{c}{8\pi} \text{Re} \left( \vec{E}_0 \times \vec{H}_2^\ast \right) \]

\[ \vec{H}_2^\ast = \frac{c}{\omega} \vec{h}_2 \times \vec{E}_0^\ast \]

\[ \vec{H}_2^\ast = \frac{c}{\omega} \vec{k}_2 \times \vec{E}_0^\ast \]

\[ \langle \mathcal{S} \rangle = \frac{c^2}{8\pi \omega} \text{Re} \left( k_2^\ast \left( \vec{E}_0 \cdot \vec{E}_0^\ast \right) - \left( \vec{E}_0 \cdot \vec{E}_0^\ast \right) \vec{E}_2^\ast \right) \]
\[ E_2^0 = E_2^0 \; \hat{E}_2 + E_2^\perp \hat{E}_\perp \]
\[ E_2^0 = E_2^0 \; \sin \theta_2 \; \hat{E}_2 + E_2^\perp \cos \theta_2 \; \hat{E}_\perp \]
\[ E_2^\perp = E_2^\perp \hat{E}_\perp \]
\[ \hat{E}_2 = \hat{k}_2 \cos \theta_2 \; \hat{E}_x - \hat{k}_2 \sin \theta_2 \; \hat{E}_y \]

with \( \sin \theta_2 = \frac{n_{\perp}}{n_{\parallel}} \) \( \sin \theta_0 = \frac{\sin \theta_e}{\sin \theta_0} \)

\[
\cos \theta_2 = i \sqrt{\frac{\sin^2 \theta_2}{\sin^2 \theta_e} - 1} = i \Omega \quad \text{Complex!} \\
\]
\[
\langle S^2 \rangle = \frac{c^2}{8 \pi \omega} \; \Re \left\{ \hat{k}_2^* \left( E_2^0 \right)^2 + E_2^\perp \right\} = \frac{c^2 \Omega^2}{8 \pi \omega} \; \Re \left\{ \left( -\Omega \cos \theta_2 \right) \hat{E}_x - \left( \sin \theta_2 \right) \hat{E}_y \right\} \left( E_2^0 \right)^2 + E_2^\perp \right\} \left( \sin^2 \theta_2 - \cos^2 \theta_2 \right) \]
\[
- \left( 2 \; E_2^0 \; \cos \theta_2 \sin \theta_2 \right) \left( E_2^0 \sin \theta_2 \hat{E}_x - E_2^0 \cos \theta_2 \hat{E}_y \right) \]
\[
+ 2 \; E_2^0 \; E_2^\perp \; \hat{E}_\perp \; \hat{E}_\perp \; \cos \theta_2 \sin \theta_2 \hat{E}_x \]

Since \( \cos \theta_2 = i \Omega \) \( \Omega \hat{E}_x \) is no power transmitted in any directions.

Reflection off of metallic surfaces:

a) Recall that we can treat conductors by just making the dielectric constant complex.

\[ \varepsilon_{ss} = \varepsilon + \frac{i \varepsilon_{ss} \omega}{\varepsilon} \quad \text{for good Conductors recall} \]

Then, we can copy all the previous results for field amplitudes (not for energies)
b) \( \perp \) incidence

\[ \begin{align*}
\vec{E}_0 & \rightarrow \vec{H}_0 \\
\vec{k}_0 & \rightarrow \vec{H}_1 \\
\vec{k}_1 & \rightarrow \vec{H}_2 \\
\vec{E}_1 & \rightarrow 0 \\
\end{align*} \]

\[ \vec{E}_0(r,t) = E_0 \hat{e}_x e^{i(k_0 r - \omega t)} \]

\[ |\vec{k}_0| = |\vec{k}_1| = \frac{\omega}{c} n \]

\[ \vec{k}_2 = \frac{\omega}{c} n \text{ eff} \hat{e}_z \]

\[ \hat{e}_x \]

\[ E_1^0 = \frac{n \text{ eff}^2 - n^2}{n \text{ eff}^2 + n^2} \]

\[ \frac{\hat{e}_x \cdot \hat{e}_z}{n \text{ eff}} = \frac{(1 + i) \frac{\omega}{\omega_0} - n}{(1 + i) \frac{\omega}{\omega_0} + n} \]

\[ E_0^0 = \frac{2n}{n \text{ eff}^2 + n^2} E_0^1 = \frac{2n}{(1 + i) \frac{\omega}{\omega_0} + n} E_0 \]

Reflection Coeff.

\[ R = \frac{\langle \hat{S}_1 \cdot \hat{C} \hat{E}_z \rangle_{(E_2)}}{\langle \hat{S}_0 \rangle_{(E_2)}} = \frac{|E_1^0|^2}{|E_0^1|^2} = \frac{\left|\frac{(1 + i) \frac{\omega}{\omega_0} - n}{(1 + i) \frac{\omega}{\omega_0} + n}\right|^2}{1 + \left(\frac{w n \text{ eff}}{c} - 1\right)^2} \]

For \( 4n \pi \sigma \gg \varepsilon \omega \) implies \( \frac{\omega}{\omega_0} \gg 1 \) is \( E_1 \gg 1 \)

\[ R \approx 1 - 2w n \text{ eff} \quad \frac{\sqrt{\omega}}{\varepsilon} \]

\[ T \approx 1 - R \]

Conclusion: Good metals have almost perfect reflection of EM waves.
For medium I the total electric field is
\[ E_0 + E_i = E_0 \hat{e}_x (e^{i(k_0 z - \omega t)} - e^{i(k_0 z + \omega t)}) \]
\[ = E_0 \hat{e}_x e^{-i\omega t} 2i \sin k_0 z \]
this is a standing wave with nodes at distances \( \frac{1}{2} m \lambda_0 = \frac{m \pi}{k_0} \) \( m = 1, 2, \ldots \)
from the interface

Aside: Verifying this result experimentally is difficult since detecting small differences from \( R = 1 \) is hard
need to use a trick: Each body

C) Oblique Incidence Refraction
\[ n_{II} \sim \frac{C}{\omega_0} (1 + i) \] complex

\[ \text{Surfaces of Constant amplitude} \]
\[ \text{Surfaces of Constant Phase} \]

\[ k_{ox} = k_{ax} \]
\[ k_{2x}^2 + k_{2z}^2 = \frac{\omega^2}{c^2} n_{II}^2 \]
\[ k_{2z} = \sqrt{\frac{\omega^2}{c^2} n_{II}^2 - k_{2x}^2} \] complex
\[ E \propto e^{i(k_{2z} - \omega t)} = e^{i(k_{ox} + Re k_{2z} - \omega t)} - Im k_{2z} \]

- Attenuation only occurs along \( z \) direction
- Surfaces of Constant Phase are determined by \( Re k_{2z} \)
\[ \tan \Theta_2 = \frac{k_{ax}}{Re k_{2z}} = \frac{k_{ax}}{Re \sqrt{\frac{\omega^2}{c^2} n_{II}^2 - k_{2x}^2}} = \frac{\sin \Theta_0}{Re \sqrt{n_{II}^2 - \sin^2 \Theta_0}} \]
If we define $\text{Re} \sqrt{N^2 - \sin^2 \theta_e} = N(\theta_e) \cos \theta_2$

$N(\theta_e) \sin \theta_2 = \sin \theta_0$

"effective Snell's law."

Compare w/ total internal reflection:

$E_2 = E_0 e^{-j k_0 x -j\omega t - jQ^2 k_2}$

where $Q = \sqrt{\frac{\sin^2 \theta_0}{\sin^2 \theta_c} - 1}$

$R = 1$ if $\theta_0 > \theta_c$

$\sin \theta_0 = n_1 \frac{n_2}{n_1}$

---

**Transmission line**

We've looked at waves travelling through space & media.

We've also looked at $E$ fields inside conductors.

Let's combine everything.

a) Joint propagation of electromagnetic waves and currents through two parallel conductors

ex: 1) Coaxial Cable

2) Parallel wires

b) Wave propagation through a transmission line:

Waves consist of currents & fields.

These are coupled. Let's use currents as our reference variable.
\[
\begin{align*}
\text{i: Current in each conductor} & \quad \text{assume equal but opposite} \\
\n\text{V: Potential difference between two conductors at position z} & \quad \text{V = -} S \text{d} \dot{E} \cdot \dot{E} \\
\text{Assume ideal Conductors: no external E field // to Conductor. For capacitor } Q = VC. \text{ We want this per unit length.} & \quad V(z) = \frac{S_e(z)}{C_e} \\
S_e(z) & \equiv \text{Charge density/ unit length} \\
C_e & \equiv \text{Capacitance/ unit length} \\
\frac{dV}{dt} = \frac{1}{C_e} \frac{dS_e}{dt} \quad \text{which by current conservation } \frac{dS_e}{dt} = -\frac{di}{dz} \quad \text{gives} \\
\frac{dV}{dt} = -\frac{1}{C_e} \frac{di}{dz} \\
V(z+\Delta z) - V(z) & \quad \text{Change in i causes change in voltage} \\
\text{hence the magnetic field } & \quad \text{ENF } E \\
\text{These two branches cancel} & \\
\frac{dV}{dt} & \quad \text{where } L_e \text{ is the inductance per unit length} \\
\frac{d^2V}{dz^2} = -L_e \frac{di}{dt} & \quad \text{wave equations} \\
\frac{d^2i}{dt^2} & = L_e C_e \frac{d^2i}{dt^2} \\
\end{align*}
\]
Since $\vec{E}, \vec{B}$ satisfy wave eq in vacuum for space between conductors & since $\vec{E} \perp \vec{B}$ & $\vec{E}$ these waves travel at speed of light $c$. So $\vec{E}$ & $\vec{B}$ are orthogonal to each other in configuration (only works for loss free lines w/ Grenell waves).

Solution: $V(z,t) = V_0 e^{i(k_2 - w)t}$

$i(z,t) = \sqrt{\frac{c_0}{\mu_0}} V_0 e^{i(k_2 - w)t}$

At $z = 0$, $V(0,t) = V_0 e^{iwt}$ is generated by AC source. Note that $i$ & $V$ are in phase! Impedance $Z_0 = \frac{V_0(0,t)}{i(0,t)} = \frac{2\pi}{\sqrt{\mu_0 c_0}}$

$Z_0 =$ Characteristic impedance of transmission line

5) Terminating the transmission line:

Connect outer ends $z = 0$ to $z = l$ to a load $\omega$ with resistance $R_{\text{load}}$

\[ Z_{\text{load}} = Z_0 \] travelling wave satisfies boundary conditions at $z = l$. This leads to no reflected wave & we say the impedance is matched.

If $Z_{\text{load}} = Z_0$, travelling wave

\[ V = V_0 e^{i(k_2 - w)t} \] & \[ i = \sqrt{\frac{c_0}{\mu_0}} V_0 e^{i(k_2 - w)t} \]

If $Z_{\text{load}} \neq Z_0$, do not satisfy the boundary condition at $z = l$. Need a reflected wave.

\[ V(z,t) = V_+ e^{i(k_2 - w)t} + V_- e^{i(k_2 - w)t} \]

\[ i(z,t) = i_+ e^{i(k_2 - w)t} + i_- e^{i(k_2 - w)t} \]
Since \( \frac{dV}{dt} = -\frac{1}{c_0} \frac{dv}{dz} \) in transmission line, 

\[
\begin{align*}
\frac{v_+}{i_+} &= \frac{\sqrt{\frac{c_0}{Z_0}}}{Z_0} = Z_0, \\
\frac{v_-}{i_-} &= -\frac{1}{\sqrt{\frac{c_0}{Z_0}}} = -Z_0.
\end{align*}
\]

Also need at \( z = l \)

\[
\frac{V(t)}{i(t)} = \frac{V_e e^{ikl} + V_m e^{-ikl}}{i_e e^{ikl} + i_m e^{-ikl}} = Z_{eo}.
\]

Solve this for HW problem

\[ Z = \frac{V(t)}{i(t)} = \text{impedance of transmission line} + \text{load seen at generator} \]

\( Z \) is a function of \( l \) & period \( \frac{\pi}{k} \)

• Amplitude of reflection Coeff \( \rho \)

\[
\rho = \frac{v_-(t)}{v_+(t)} = \frac{-i_-}{i_+} = \frac{Z_{load} - Z_0}{Z_{load} + Z_0}.
\]

Power reflection Coeff \( \eta \)

\[
\eta = \left| \frac{Z_{load} - Z_0}{Z_{load} + Z_0} \right|^2.
\]

Limiting Cases:

* \( Z_{load} = R \gg Z_0 \) (open end)

\[
Z = \frac{Z_0}{-i \tan k l} \quad R = 1
\]

* \( Z_{load} = R \ll Z_0 \) (short)

\[
Z = -i \frac{Z_0}{\tan k l} \quad R \approx 1
\]

* only if \( Z_{load} = Z_0 \) close \( R \) differ appreciably from unity