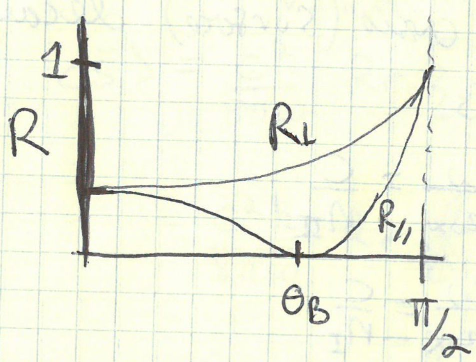
$\vec{E}_0 \parallel$ plane of incidence

$$R_{\parallel} = \frac{\tan^2(\theta_2 - \theta_0)}{\tan^2(\theta_2 + \theta_0)} \qquad T_{\parallel} = 1 - R_{\parallel}$$

Define degree of polarization of reflected wave:

$$P = \frac{R_{\perp} - R_{\parallel}}{R_{\perp} + R_{\parallel}} \qquad P = 0 \text{ if } \theta_0 = 0, \frac{\pi}{2} \qquad P > 0 \text{ otherwise}$$

$$P(\theta_B) = 1$$



total internal Reflection

1) what if $n_I > n_{II}$?

Then $\sin\theta_2 = \frac{n_I}{n_{II}} \sin\theta_0$ has no solution if

$$\theta_0 > \underbrace{\arcsin \frac{n_{II}}{n_I}}_{\theta_c}$$

for $\theta_0 > \theta_c$ no transmitted wave is possible.
Reflection is perfect/total.

2) Lets work this out in greater detail:

$$\vec{E}_0 = \vec{E}_0^0 e^{i(k_0 r - \omega t)}$$

$$\vec{E}_1 = \vec{E}_1^0 e^{i(k_1 r - \omega t)}$$

$$\vec{E}_2 = E_2^0 e^{i(k_2 r - \omega t)}$$

k_j in xy plane $j = 0, 1, 2$

we must have

$$k_{0x} = k_{1x} = k_{2x}$$

what is k_{2z}?

$$k_2^2 = \underbrace{\left(\frac{\omega}{c}\right)^2 n_{II}^2} = k_{2x}^2 + k_{2z}^2 = k_{0x}^2 + k_{2z}^2 = k_0^2 \sin^2\theta_0 + k_{2z}^2 = \underbrace{\left(\frac{\omega}{c}\right)^2 n_I^2 \sin^2\theta_0 + k_{2z}^2}$$

$$k_{2z} = \frac{\omega}{c}\sqrt{n_{II}^2 - n_I^2 \sin^2\theta_0} = \underbrace{\frac{\omega}{c} n_{II}}_{k_2} \sqrt{1 - \frac{\sin^2\theta_0}{\sin^2\theta_c}} \qquad \sin\theta_c = \frac{n_{II}}{n_I}$$

if $\theta_0 > \theta_c$: $k_{2z} = i\frac{\omega}{c} n_{II} \sqrt{\frac{\sin^2\theta_0}{\sin^2\theta_c} - 1} = i k_2 Q$

$$\vec{E}_2 = \vec{E}_2^0 e^{ik_0 x - i\omega t - Qzk_2}$$

wave propagation along x axis (surface) decaying into medium along z axis

Phase velocity at $\theta_0 = \theta_c$: $\frac{\omega}{k_{0x}} = \frac{c}{n_{II}}$

at $\theta_0 = \frac{\pi}{2}$: $\frac{\omega}{k_{0x}} = \frac{c}{n_I}$

Decay length $\frac{1}{\operatorname{Im} k_{2z}} = \frac{1}{Qk_2}$

Can show at Home that $R = \frac{|E_1^0|^2}{|E_0^0|^2} = 1$

3) Energy flow in medium II

(Good excersize in complex analysis. So good that both Heald & Marion & Jackson don't get it right)

$$\langle \vec{S} \rangle = \frac{c}{8\pi} \operatorname{Re} \vec{E}_2^0 \times \vec{H}_2^{0*}$$

use $\vec{H}_2^0 = \frac{c}{\omega} \vec{k}_2 \times \vec{E}_2^0 \qquad H_2^* = \frac{c}{\omega} \vec{k}^* \times \vec{E}_2^{0*}$

$$\langle \vec{S} \rangle = \frac{c^2}{8\pi\omega} \operatorname{Re} \left(k_2^* (\vec{E}_2^0 \cdot \vec{E}_2^{0*}) - (\vec{E}_2^0 \cdot \vec{k}_2^*) \vec{E}_2^{0*}\right)$$

use $\vec{E}_2^\circ = E^\circ_{2\parallel} + E^\circ_{2\perp}$

$$\vec{E}^\circ_{2\parallel} = E^\circ_{2\parallel} \sin\theta_2 \hat{e}_z + E^\circ_{2\parallel} \cos\theta_2 \hat{e}_x$$

$$\vec{E}^\circ_{2\perp} = E_{2\perp} \hat{e}_y$$

$$\vec{k}_2 = k_2 \cos\theta_2 \hat{e}_z - k_2 \sin\theta_2 \hat{e}_x$$

with $\sin\theta_2 = \frac{n_I}{n_{II}} \sin\theta_0 = \frac{\sin\theta_0}{\sin\theta_c}$

$\cos\theta_2 = i\sqrt{\frac{\sin^2\theta_0}{\sin^2\theta_c} - 1} = iQ$ Complex!

$$\langle \vec{S} \rangle = \frac{c^2}{8\pi\omega} \mathrm{Re}\left\{ \vec{k}_2^* \left(|\vec{E}^\circ_{2\parallel}|^2 + |\vec{E}^\circ_{2\perp}|^2 \right) - \left(\vec{E}^\circ_{2\parallel} \cdot \vec{k}_2^* \right)\left(\vec{E}^{\circ *}_{2\perp} + \vec{E}^{\circ *}_{2\parallel} \right) \right\}$$

$$= \frac{c^2 k_2}{8\pi\omega} \mathrm{Re}\left\{ \left((-\cos\theta_2)\hat{e}_z - (\sin\theta_2)\hat{e}_x \right)\left(\underbrace{|E^\circ_{2\perp}|^2}_{|\vec{E}^\circ_{2\perp}|^2} + \underbrace{|E^\circ_{2\parallel}|^2 (\sin^2\theta_2 - \cos^2\theta_2)}_{|\vec{E}^\circ_{2\parallel}|^2} \right) \right.$$
$$\left. - \left(-2 E^\circ_{2\parallel} \cos\theta_2 \sin\theta_2 \right)\left(E^{\circ *}_{2\parallel} \sin\theta_2 \hat{e}_z - E^{\circ *}_{2\parallel} \cos\theta_2 \hat{e}_x + E^{\circ *}_{2\perp} \hat{e}_y \right) \right\}$$

used $\sin^2\theta_2 + \cos^2\theta_2 = 1$ to simplify

$$= \frac{c^2 k_2}{8\pi\omega} \mathrm{Re}\left\{ -\cos\theta_2 \hat{e}_z \left(|E^\circ_{2\perp}|^2 - |E^\circ_{2\parallel}|^2 \right) - \sin\theta_2 \hat{e}_x \left(|E^\circ_{2\perp}|^2 - |E^\circ_{2\parallel}|^2 \right) + 2 E^\circ_{2\parallel} E^{\circ *}_{2\perp} \hat{e}_y \cos\theta_2 \sin\theta_2 \right\}$$

Since $\cos\theta_2 = iQ$ there is no power transmitted in z or y directions

---

Reflection off of metalic surfaces:

a) Recall that we can treat conductors by just making the dielectric constant complex

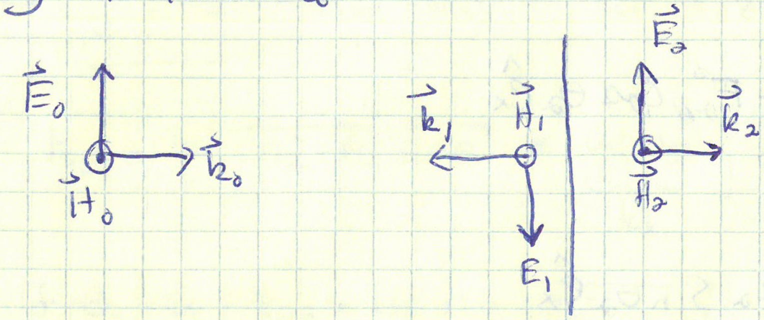
$\epsilon_{eff} = \epsilon + \frac{4\pi i \sigma}{\omega}$ $\quad n_{eff} = \sqrt{\epsilon_{eff}}$ for good conductors recall (Limiting Case)

$n_{eff} = \frac{c}{\omega\delta}(1+i)$ $\quad \delta = \frac{c}{\sqrt{2\pi\sigma\omega}}$

Complex index of refraction $\quad$ Skin depth

Then, we can copy all the previous results for field amplitudes (not for energies)

b) $\perp$ incidence



$$\vec{E}_0(\vec{r},t) = E_0^\circ \hat{e}_x e^{i(\vec{k}_0\cdot\vec{r}-\omega t)} \qquad |\vec{k}_0| = |\vec{k}_1| = \frac{\omega}{c} n_I$$

$$\vec{E}_1(\vec{r},t) = -E_1^\circ \hat{e}_x e^{i(\vec{k}_1\cdot\vec{r}-\omega t)} \qquad \vec{k}_2 = \frac{\omega}{c} n_{II\,eff}\,\hat{e}_z$$

$$\vec{E}_2(\vec{r},t) = E_2^\circ \hat{e}_x e^{i(k_2 r-\omega t)}$$

$$E_1^\circ = \frac{n_{II\,eff}-n_I}{n_{II\,eff}+n_I} E_0^\circ = \frac{(1+i)\frac{c}{\omega\delta}-n_I}{(1+i)\frac{c}{\omega\delta}+n_I}$$

$$E_2^\circ = \frac{2n_I}{n_{II\,eff}+n_I} E_0^\circ = \frac{2n_I}{(1+i)\frac{c}{\omega\delta}+n_I} E_0^\circ$$

Reflection Coeff.

$$R = \frac{\langle \vec{S}_1\rangle(-\hat{e}_z)}{\langle \vec{S}_0\rangle(e_z)} = \frac{|E_1^\circ|^2}{|E_0^\circ|^2} = \left|\frac{(1+i)\frac{c}{\omega\delta}-n_I}{(1+i)\frac{c}{\omega\delta}+n_I}\right|^2 = \frac{1+\left(\frac{\omega\delta n_I}{c}-1\right)^2}{1+\left(\frac{\omega\delta n_I}{c}+1\right)^2}$$

for $4\pi\sigma \gg \epsilon_{II}\omega$ implies $\frac{c}{\omega\delta} \gg 1$ is $\epsilon_{II} > 1$

$$R \simeq 1 - \frac{2\omega\delta n_I}{c} = 1 - 2n_I\sqrt{\frac{\omega}{2\pi\sigma}}$$

$$T \simeq 1-R \approx \frac{2\omega\delta n_I}{c} = 2n_I\sqrt{\frac{\omega}{2\pi\sigma}}$$

This transmitted energy will be absorbed by Conductor with in length δ

Conclusion: Good metals have almost perfect reflection of EM waves

For medium I, the total electric field is

$$\vec{E}_0 + \vec{E}_1 \approx E_0^0 \hat{e}_x \left(e^{i(k_0 z - \omega t)} - e^{i(k_0 z - \omega t)}\right)$$

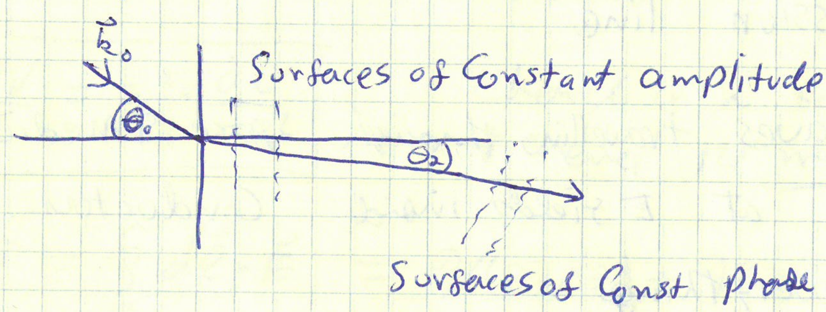
$$= E_0^0 \hat{e}_x e^{-i\omega t} \; 2i \sin k_0 z$$

this is a standing wave with nodes at distances $\frac{1}{2} m \lambda_0 = \frac{m\pi}{k_0}$ $m = 1, 2, \ldots$ from the interface

Aside:

~~Verifying this result experimentally is difficult since detecting small differences from $R=1$ is hard need to use a trick: Each body~~

c) Oblique Incidence Refraction $\quad n_{II} \sim \frac{c}{\omega\delta}(1+i)$ complex



$$k_{0x} = k_{2x}$$

$$k_{2x}^2 + k_{2z}^2 = \frac{\omega^2}{c^2} n_{II}^2$$

$$k_{2z} = \sqrt{\frac{\omega^2}{c^2} n_{II}^2 - k_{2x}^2} = \text{Complex } \#$$

$$\vec{E} \propto e^{i(\vec{k}_2 \cdot \vec{r} - \omega t)} = e^{i(k_{2x} x + \operatorname{Re} k_{2z} z - \omega t) - \operatorname{Im} k_{2z} z}$$

- Attenuation only occurs along z direction
- Surfaces of Constant Phase are determined by $\operatorname{Re} k_{2z} z$

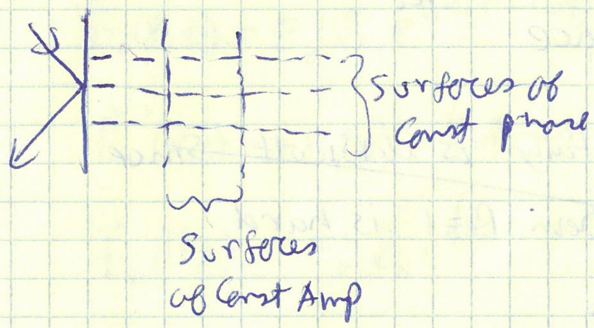
$$\tan \theta_2 = \frac{k_{2x}}{\operatorname{Re} k_{2z}} = \frac{k_{0x}}{\operatorname{Re}\sqrt{k_0^2 n_{II}^2 - k_{0x}^2}} = \frac{\sin\theta_0}{\operatorname{Re}\sqrt{n_{II}^2 - \sin^2\theta_0}}$$

If we define $\mathrm{Re}\sqrt{n_{II}^2 - \sin^2\theta_0} \equiv \underbrace{N(\theta_0)}_{\text{effective index of refraction}} \cos\theta_2$

$N(\theta_0)\sin\theta_2 = \sin\theta_0$

"effective Snell's law!"

Compare w/ total internal reflection:



$$\vec{E}_2 = E_2^0 e^{i k_0 x - i\omega t - Q z k_2}$$

where $Q = \sqrt{\dfrac{\sin^2\theta_0}{\sin^2\theta_c} - 1}$

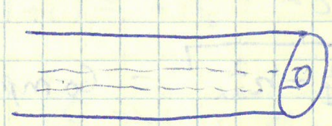
$R = 1$ if $\theta_0 > \theta_c$ $\quad \sin\theta_c = \dfrac{n_{II}}{n_I}$

Transmission line

We've looked at waves travelling through space & media
we've also looked at E fields inside conductors.
Lets combine everything

a) Joint propagation of electromagnetic waves and currents through two parallel conductors

ex: ① Coaxial Cable

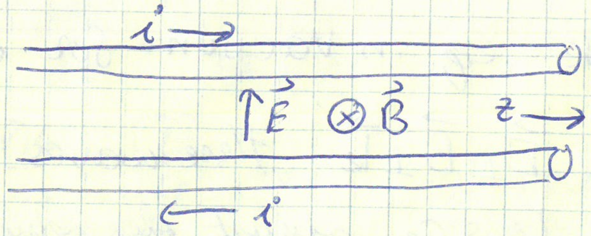


② Parallel wires

b) wave propagation through a transmission line:

waves consist of currents & fields

These are coupled ~~so~~ lets use currents as our reference variables



i: Current in each conductor assume equal & opposite

$\mathcal{V}$: Potential difference between two conductors at position z_0

$$\mathcal{V} = -\int d\vec{l}\cdot\vec{E}$$

assume ideal conductors: no external E field // to conductor. For capacitors $Q = \mathcal{V}C$. We want this per unit length.

$$\mathcal{V}(z) = \rho_\ell(z)/C_\ell$$

$\rho_\ell(z) \equiv$ Charge density/unit length

$C_\ell \equiv$ Capacitance/unit length

The equations:

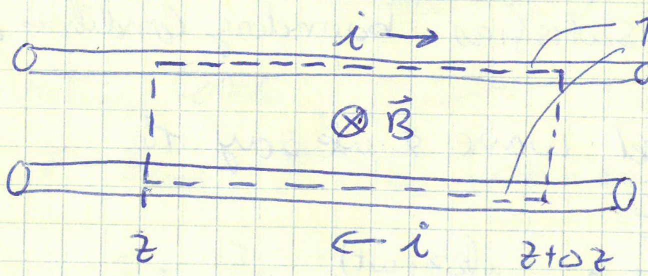
$$\frac{\partial \mathcal{V}}{\partial t} = \frac{1}{C_\ell}\frac{\partial \rho_\ell}{\partial t}$$

which by current conservation $\frac{\partial \rho_\ell}{\partial t} = -\frac{\partial i}{\partial z}$ gives

$$\boxed{\frac{\partial \mathcal{V}}{\partial t} = -\frac{1}{C_\ell}\frac{\partial i}{\partial z}}$$

Change in i causes change in voltage & hence the magnetic field & EMF $\mathcal{E}$

$$\mathcal{E} = \oint_\Gamma d\vec{l}\cdot\vec{E} = \mathcal{V}(z+\Delta z) - \mathcal{V}(z)$$



These two branches cancel

$$\mathcal{E} = -\frac{1}{c}\frac{d\Phi_m}{dt} \propto \frac{di}{dt}$$

$$\mathcal{E} = -L_\ell \frac{di}{dt}\Delta z$$

$$\therefore \frac{\partial \mathcal{V}}{\partial z} = -L_\ell \frac{di}{dt}$$

where L_ℓ is the inductance per unit length

$$\frac{\partial^2 \mathcal{V}}{\partial z^2} = -L_\ell \frac{\partial}{\partial t}\frac{\partial i}{\partial z} = L_\ell C_\ell \frac{\partial^2 \mathcal{V}}{\partial t^2}$$

or

$$\frac{\partial^2 i}{\partial z^2} = L_\ell C_\ell \frac{\partial^2 i}{\partial t^2}$$

wave equations

wave speed $c = \frac{1}{\sqrt{L_\ell C_\ell}}$

Since $\vec{E}, \vec{B}$ satisfy wave eq in vacuum for space between conductors & since $\vec{E} \perp \vec{B} \perp \vec{k}$ these waves travel at speed of light c so L_ℓ & C_ℓ adjust ~~dependin~~ on configuration (only works for loss free lines w/ const cross section)

Solution: $V(z,t) = V_0 e^{i(kz-\omega t)}$ $\quad \frac{\omega}{k} = c$

$$i(z,t) = \sqrt{\frac{C_\ell}{L_\ell}}\, V_0 e^{i(kz-\omega t)}$$

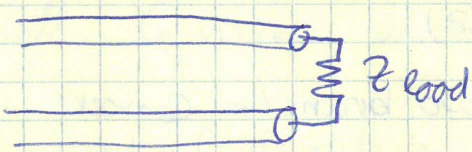
at $z=0$ $\quad V(0,t) = V_0 e^{i\omega t}$ is generated by AC source

Note that i & V are in phase! Impedence $Z_0 = \frac{V(0,t)}{i(0,t)} = \sqrt{\frac{L_\ell}{C_\ell}}$

$Z_0 \equiv$ Characteristic impedence of transmission line

c) Terminating the transmission line:

Connect other ends of transmission line at $z=\ell$ to a load w/ resistence Z_{load}



If $Z_{load} = Z_0$: travelling wave satisfies boundary conditions at $z=\ell$

This leads to no reflected wave & we say the "impedence is matched"

If $Z_{load} \neq Z_0$ $\quad V = V_0 e^{i(kz-\omega t)}$ &

$$i = \sqrt{\frac{C_\ell}{L_\ell}}\, V_0 e^{i(kz-\omega t)}$$

do not satisfy the boundary condition at $z=\ell$

Need a reflected wave.

$$V(z,t) = V_+ e^{i(kz-\omega t)} + V_- e^{i(-kz-\omega t)}$$

$$i(z,t) = i_+ e^{i(kz-\omega t)} + i_- e^{i(-kz-\omega t)}$$

Since $\frac{\partial V}{\partial t} = -\frac{1}{c_\ell}\frac{\partial i}{\partial z}$ in transmission line $\quad -i\omega V_\pm = \frac{-i(\pm k)}{c_\ell}\, i_\pm$

$$\Rightarrow \begin{cases} \dfrac{V_+}{i_+} = \sqrt{\dfrac{L_\ell}{c_\ell}} = Z_0 \\[2ex] \dfrac{V_-}{i_-} = -\sqrt{\dfrac{L_\ell}{c_\ell}} = -Z_0 \end{cases}$$

also need at $z=\ell$ $\quad \dfrac{V(\ell,t)}{i(\ell,t)} = \dfrac{V_+ e^{ik\ell} + V_- e^{-ik\ell}}{i_+ e^{ik\ell} + i_- e^{-ik\ell}} = Z_{load}$

solve this for HW problem

Sub in for V_+, V_-, i_+, i_- using previous formulas (prob 7.3)

$Z = \dfrac{V(0,t)}{i(0,t)}$ = impedence of transmission line + load seen at generator $= \dfrac{V_+ + V_-}{i_+ + i_-} = Z_0 \dfrac{Z_{load} - i Z_0 \tan k\ell}{Z_0 - i Z_{load} \tan k\ell}$

Z is a function of ℓ & period $\frac{\pi}{k}$

• amplitude of reflection Coeff

$$\Gamma \equiv \frac{V_-(\ell,t)}{V_+(\ell,t)} = \frac{-i_-(\ell,t)}{i_+(\ell,t)} = \frac{Z_{load} - Z_0}{Z_{load} + Z_0}$$

Power reflection Coeff

$$R = |\Gamma|^2 = \left|\frac{Z_{load} - Z_0}{Z_{load} + Z_0}\right|^2$$

Limiting Cases:

resistor

☆ $Z_{load} = R \gg Z_0$ (open end)

$Z = \dfrac{Z_0}{-i \tan k\ell} \qquad R = 1$

resistor

* $Z_{load} = R \ll Z_0$ (Short)

$Z = -i Z_0 \tan k\ell \qquad R \approx 1$

* only if $Z_{load} = Z_0$ does R differ appreciably from unity