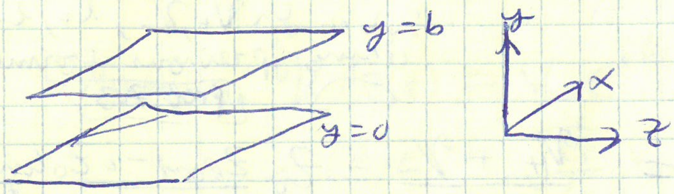
Case 2: Waves between Conducting Planes

a) Use the boundary conditions for a perfect conductor
$\vec{E}_{\parallel}=0$ $\vec{B}_{\perp}=0$ & $\mu=\epsilon=1$ between planes



One solution is to have Transverse Electromagnetic waves (TEM) waves

Here $\vec{E}(\vec{r},t) = E^0 \hat{e}_y e^{i(kz-\omega t)}$
$\vec{B}(\vec{r},t) = -B^0 \hat{e}_x e^{i(kz-\omega t)}$

However there are also other solutions in which $\vec{E}$ is $\perp$ to $\vec{k}$ (TE) or $\vec{B} \perp \vec{k}$ (TM) but not both.

In hollow conductors (w/ walls on all sides) TE & TM are allowed. TEM are not.

Lets construct a solution with $\vec{E} \parallel \hat{e}_x$ & $\vec{B}$ in arbitrary direction

$$\vec{E}(\vec{r},t) = \hat{e}_x E_0^{\circ} e^{i(k\cdot r-\omega t)} \quad \text{with } \vec{k} = k_y \hat{e}_y + k_z \hat{e}_z$$

also add wave with $\vec{k} = -k_y \hat{e}_y + k_z \hat{e}_z$

$$\vec{E}(\vec{r},t) = \hat{e}_x E_0^{\circ} e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \hat{e}_x E_1^{\circ} e^{i(-k_y y + k_z z - \omega t)}$$

B.C.: Choose $E_1^{\circ} = -E_0^{\circ}$ to ensure $\vec{E}=0$ at $y=0$

Then $\vec{E}(\vec{r},t) = \hat{e}_x E_0^{\circ} e^{i(k_z z-\omega t)}\, 2i \sin(k_y y)$

also require $\vec{E}=0$ if $y=b$

$k_y b = n\pi \quad n=1,2,\ldots$ n is the mode #

$k_y b = 0$ gives $\vec{E}=0$ everywhere (trivial solution)

Solution: $\vec{E}(\vec{r},t) = \hat{e}_x E_0^\circ \left(-2i \sin \frac{n\pi}{b}\right) e^{i(k_z z - \omega t)}$

now only a function of z

$k = -k_y \hat{e}_y + k_z \hat{e}_z \Rightarrow k_z^2 + \left(\frac{n\pi}{b}\right)^2 = \left(\frac{\omega}{c}\right)^2$

$k_z = \sqrt{\frac{\omega^2}{c^2} - \left(\frac{n\pi}{b}\right)^2}$ dispersion relation

Magnetic Field:

For travelling wave: $\vec{B} = \hat{e}_k \times \vec{E} = \frac{\vec{k}\times\vec{E}}{k}$

$$\begin{vmatrix} e_x & e_y & e_z \\ 0 & k_y & k_z \\ E_x & 0 & 0 \end{vmatrix} \longrightarrow \vec{B}(\vec{r},t) = \left(-\frac{\hat{e}_z k_y}{k} + \frac{\hat{e}_y k_z}{k}\right) E_0^\circ e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\begin{vmatrix} e_x & e_y & e_z \\ 0 & -k_y & k_z \\ E_x & 0 & 0 \end{vmatrix} \Rightarrow -\left(\frac{\hat{e}_z k_y}{k} + \frac{\hat{e}_y k_z}{k}\right) E_0^\circ e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= -\frac{k_y}{k} E_0^\circ \hat{e}_z \, 2\cos(k_y y)\, e^{i(k_z z - \omega t)}$$

$$+ \frac{k_z}{k} E_0^\circ \hat{e}_y \, 2i \sin(k_y y)\, e^{i(k_z z - \omega t)}$$

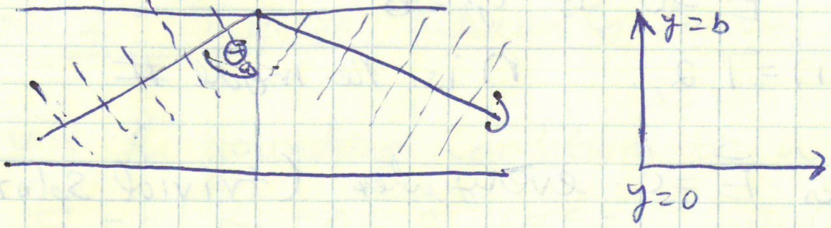
$\vec{B}_\perp = 0$ if $y=0,b$ since $k_y = \frac{n\pi}{b}$

B_y

Wave consists of two superimposed plane waves moving with $\vec{k} = \pm k_y \hat{e}_y + k_z \hat{e}_z$

with angle θ_0 defined by $\cos\theta_0 = \frac{n\pi}{kb}$ w/respect to y axis

$\lambda_y = \lambda_0 / \sin\theta_0$ effective wave length along z direction



These waves reflect specularly off the perfectly Conducting planes

→ Standing wave in y direction
travelling wave in z direction w/ $k_z = \sqrt{\frac{\omega^2}{c^2} - \left(\frac{n\pi}{b}\right)^2}$

Note: for propagating wave k_z must be real
$k^2 = \frac{\omega^2}{c^2} > \left(\frac{n\pi}{b}\right)^2$ or equivalently

$\lambda = \frac{2\pi}{k} < \frac{2b}{n} \equiv \lambda_c$ the cut off wavelength

$\nu = \frac{\omega}{2\pi} > \frac{nc}{2b} \equiv \nu_c = \frac{\omega_c}{2\pi}$ ω_c, ν_c are the cut off frequencies

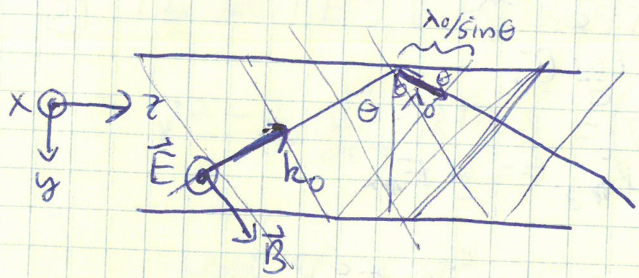
if $\lambda > \lambda_c$ or $\omega < \omega_c$ wave is damped in z direction

Phase & group velocity: $u_{ph} = \frac{\omega}{k_z} = \frac{c}{\sin\theta_0}$
is the velocity at which the phase fronts move through the wave guide

Note: The actual travelling waves that build the TE mode are travelling at speed c
· The z component of their velocity is $u_{gr} = c\sin\theta_0 = \frac{\omega k_z}{k^2}$

this group velocity indicates how fast information & energy travels $u_{gr} = \frac{d\omega}{dk_z} = \frac{k_z c}{\sqrt{k_z^2 + (n\pi/b)^2}} = c\sin\theta_0$ $\omega = c\sqrt{k_z^2 + \left(\frac{n\pi}{b}\right)^2}$

Example to work through

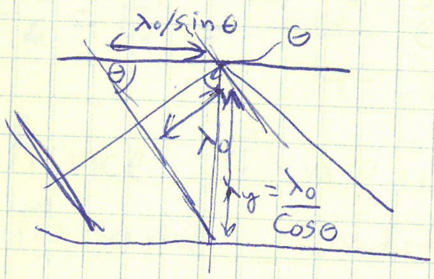


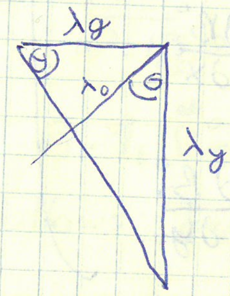
k_0 decomposes into components like any other vector.

$$k_z = k_0 \sin\theta \qquad\qquad k_y = k_0 \cos\theta$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\lambda_z = \frac{2\pi}{k_0 \sin\theta} = \frac{\lambda_0}{\sin\theta} \qquad\qquad \lambda_y = \frac{\lambda_0}{\cos\theta}$$





Waves in Hollow Conductors.

arbitrary shape translational invariance in z.

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{Bmatrix}\vec{E}\\ \vec{B}\end{Bmatrix} = 0$$

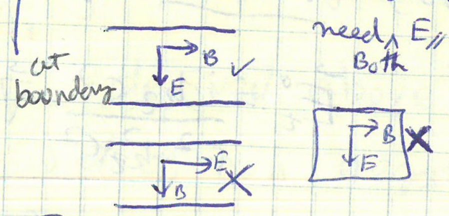
$$\vec{E} = \vec{E}_0(x,y)\, e^{i(k_z z - \omega t)}$$

$$\vec{B} = \vec{B}_0(x,y)\, e^{i(k_z z - \omega t)}$$

$E_{\parallel} = B_{\perp} = 0$ at boundaries

general result:

No TEM waves possible inside a single connected hollow conductor



need both $E_{\parallel} = 0$ & $B_{\perp} = 0$ for TEM waves

For more rigorous proof see book.

1) TE waves

Look for solution w/ $E_{z0} = 0$ but not necessarily $B_{z0} = 0$

$$\vec{\nabla}\times\vec{E} = \frac{i\omega}{c}\vec{B} \Rightarrow \begin{cases} -ik_z E_y^\circ = i\frac{\omega}{c} B_x^\circ \\ ik_z E_x^\circ = \frac{i\omega}{c} B_y^\circ \end{cases}$$

$$\vec{\nabla}\times\vec{B} = -\frac{i\omega}{c}\vec{E} \Rightarrow \begin{cases} \frac{\partial B_z^\circ}{\partial y} - ik_z B_y^\circ = -i\frac{\omega}{c} E_x^\circ \\ ik_z B_x^\circ - \frac{\partial}{\partial x} B_z^\circ = -i\frac{\omega}{c} E_y^\circ \end{cases}$$

Combine equations

$$\left.\begin{aligned} E_x^\circ &= \frac{i\omega/c}{\left(\frac{\omega}{c}\right)^2 - k_z^2}\frac{\partial B_z^\circ}{\partial y} \\ E_y^\circ &= \frac{-i\omega/c}{\left(\frac{\omega}{c}\right)^2 - k_z^2}\frac{\partial B_z^\circ}{\partial x} \\ B_x^\circ &= \frac{ik_z}{\left(\frac{\omega}{c}\right)^2 - k_z^2}\frac{\partial B_z^\circ}{\partial x} \\ B_y^\circ &= \frac{ik_z}{\left(\frac{\omega}{c}\right)^2 - k_z^2}\frac{\partial B_z^\circ}{\partial y} \end{aligned}\right\}$$

Note: denominator is not zero since $k_z \neq \frac{\omega}{c}$ for a wave that is not a TEM wave.

Can rewrite these in a two dimensional vector notation:

Define $E_t^\circ = E_x^\circ \hat{e}_x + E_y^\circ \hat{e}_y$

$B_t^\circ = B_x^\circ \hat{e}_x + B_y^\circ \hat{e}_y$

$\nabla_t = \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y}$

then: $\vec{B}_t^\circ = \frac{ik_z c^2}{\omega^2 - k_z^2 c^2}\vec{\nabla}_t B_z^\circ = k_z\frac{c}{\omega}\left(\hat{e}_z \times \vec{E}_t^\circ\right)$

Conclusion: for TE waves B_z° completely determines the fields.

2) Similarly for TM waves E_z° completely determines the fields

$$\vec{E}_t^\circ = \frac{ik_z c^2}{\omega^2 - k_z^2 c^2}\vec{\nabla}_t E_z^\circ = -\frac{k_z c}{\omega}\left(\hat{e}_z \times \vec{B}_t^\circ\right)$$

$\frac{k_z}{k_0}$ this time since B field is transverse.

3) B_z° for TE, & E_z° for TM satisfy the wave eq.:

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\begin{Bmatrix} B_z^\circ \\ E_z^\circ \end{Bmatrix} = \left(\nabla_t^2 + \left(\frac{\omega^2}{c^2} - k_z^2\right)\right)\begin{Bmatrix} B_z^\circ \\ E_z^\circ \end{Bmatrix} = 0 \quad \begin{Bmatrix} TE \\ TM \end{Bmatrix}$$

Used $\nabla^2 = \nabla_t^2 + \frac{\partial^2}{\partial z^2} = \nabla_t^2 - k_z^2$ where $\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Boundary Conditions: let $\hat{n}$ be a vector $\perp$ to boundary so that it is in $x-y$ plane

For TE mode $\left.\begin{aligned} \vec{E}\times\hat{n} &= 0 \\ \vec{B}\cdot\hat{n} &= 0 \end{aligned}\right\}$ at boundary

$\hat{e}_z \perp \hat{n}$ so

$\underbrace{\hat{e}_z \times \vec{\nabla}_t B_z^\circ}_{} \times \underbrace{n}_{} = 0$

must be //

$\vec{\nabla}_t B_z^\circ$ must be $\perp$ to $\hat{n}$

use $\vec{E}^\circ = \vec{E}_t^\circ = -\frac{\omega}{k_z c}\,\vec{e}_z \times \vec{B}_t^\circ = \frac{-i\omega c}{\omega^2 - k_z^2 c^2}\,\hat{e}_z \times \vec{\nabla}_t B_z^\circ$

hence $\vec{E}\times\hat{n} = 0$ implies $\hat{n}\cdot\vec{\nabla}_t B_z^\circ = 0$ at boundary

The same condition follows from $\vec{B}\cdot\vec{n} = 0$ at boundary.

For TM modes

$E_z = 0$ at boundary (since $E_{\parallel}$ must be zero)

This implies that the derivative of E_z along boundary is also zero.

$\hat{n}\times\vec{\nabla}_t E_z^\circ = 0$ so that $\vec{E}_{\parallel}$ & $\vec{B}_{\perp}$ are zero too (using 2) above)

4) In general the Eq:

$$\left(\nabla_t^2 + \left(\left(\frac{\omega}{c}\right)^2 - k_z^2\right)\right)\begin{Bmatrix} E_z^\circ \\ B_z^\circ \end{Bmatrix} = 0$$

with boundary conditions derived above has nontrivial solutions for a discrete set of non negative eigenvalues only.

$$\left(\frac{\omega}{c}\right)^2 - k_z^2 = K_n^2 \qquad n = 1, 2, 3, \ldots$$

hence $k_z = \sqrt{\left(\frac{\omega}{c}\right)^2 - K_n^2}$

- Cut off frequency for mode n is $\omega_c = K_n c$
- group velocity $u_g = \frac{d\omega}{dk_z} = c\sqrt{1-\frac{K_n^2 c^2}{\omega^2}} = c\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}$

$u_g \to 0$ for $\omega \downarrow \omega_c$

$u_g \to c$ for $\frac{\omega}{\omega_c} \to \infty$

5) Example: Rectangular wave guide, TE

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+K^2\right)B_z^0 = 0$$

$$\frac{\partial B_z}{\partial x} = 0 \quad \text{at} \quad x = 0, a$$

$$\frac{\partial B_z}{\partial y} = 0 \quad \text{at} \quad y = 0, b$$

Solution: $B_z^0 = B^0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$

$$K_{mn}^2 = \left(\frac{\omega}{c}\right)^2 - k_z^2 = \pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$$

mode w/ m,n is called **$TE_{m,n}$** mode

(TE_{00} mode does not exist)

if $a>b$ mode w/ lowest cut off freq

$\omega_c = K_{mn} c$ is TE_{10} "dominant mode"

In practice Engineers choose a, b such that the dominant mode is the only mode at the desired freq.

In dominant mode:

$$E_y^0 = \frac{i\omega a}{\pi c} B^0 \sin\frac{\pi x}{a} \qquad B_z^0 = B^0 \cos\frac{\pi x}{a}$$

$$B_x^0 = \frac{-ik_z a}{\pi} B^0 \sin\frac{\pi x}{a} \qquad E_x^0 = B_y^0 = E_z^0 = 0$$

Energy flow

$$\langle \vec{S} \rangle = \frac{c}{8\pi} \mathrm{Re}\left(\vec{E} \times \vec{B}^*\right)$$

For dominant mode

$$\langle \vec{S} \rangle_{10} = \frac{c}{8\pi} \mathrm{Re}\left(E_y^0 \underbrace{B_z^{0*}}_{\text{imaginary}} \hat{e}_x - E_y^0 B_x^{0*} \hat{e}_z \right)$$

$$= \frac{c}{8\pi} \hat{e}_z \left(\left|\frac{a B^0}{\pi}\right|^2 \frac{\omega}{c} k_z \right) \underbrace{\sin^2 \frac{\pi x}{a}}_{\text{ave out to } \frac{1}{2}}$$

total transmitted power in z direction

$$P_{10} = \frac{c}{16\pi} \left|\frac{a B^0}{\pi}\right|^2 \frac{\omega k_z}{c} ab$$

using $E^0 = \frac{i \omega a}{\pi c} B^0$ & $k_z = \frac{\omega}{c}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}$

$$P_{10} = \frac{c |E^0|^2}{16\pi} ab \sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}$$

Same story for TM modes now

$$E_z^0(x,y) = E^0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad m, n \geq 1$$

Lowest mode is TM_{11}

$$(\omega_c)_{mn} = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

Optical fibers / cylinders

Optical Fibers & Cylindrical wave guides:

Math Aside: Bessel functions

recall $J_n(kr) \simeq \sqrt{\frac{2}{\pi kr}} \cos\left(kr - \frac{n\pi}{2} - \frac{\pi}{4}\right)$
$N_n(kr) \sim \sqrt{\frac{2}{\pi kr}} \sin\left(kr - \frac{n\pi}{2} - \frac{\pi}{4}\right)$ } in large r limit

or $N_n(z) = \dfrac{\cos n\pi\, J_n(z) - J_{-n}(z)}{\sin n\pi}$

Holds For Complex arguments

modified Bessel function of the first kind:

$$I_\nu(z) = e^{-\pi i \nu/2} J_\nu(iz)$$

modified Bessel function of second kind

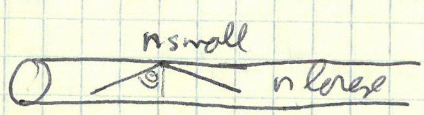
$$K_\nu(z) = \frac{\pi}{2\sin\pi\nu}\left(I_{-\nu}(z) - I_\nu(z)\right)$$

$$K_\nu(z) \approx \sqrt{\frac{\pi}{2z}}\, e^{-z} \quad \text{if } z \gg |\nu|$$

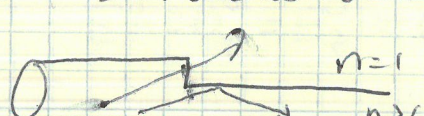
Optical Fibers

Metal wave guides have losses: every reflection, some intensity is lost. They are also easy to break.

Optical fibers use total internal reflection



Problems:
- * Need a minimum θ for total internal reflection
- * excitation of evanescent waves can lead to losses
- * Complicated Boundary Conditions at interface
  ex: reflection has π phase shift depending on θ
- * Practical difficulties - Scratches on surface causes Light to leak



Solution protect surface by adding another dielectric cladding

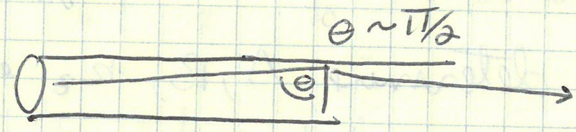
$n=1$

$n_2>1$ cladding

$n_1>n_2$ core.

Now, light stays in core, damage to outer surface has no consequences if cladding is thicker than decay length for evanescent waves

to simplify theory, assume n_2 is just a bit smaller than n_1



waves are almost TEM waves

For each field component $E_x, \dots B_z$ labled ψ we have

$$\left(\nabla^2 - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi = 0 \qquad \psi = \psi_0(r)\, e^{i(k_z z - \omega t)}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial}{\partial r} + \underbrace{\left[\left(\frac{n\omega}{c}\right)^2 - k_z^2\right]}_{k_c^2}\right)\psi = 0$$

Bessel eq for $x\,y\,z$ components. kind of wierd

* for $r<a$ (core)

  need $k_c^2 > 0$ so that $\psi_0(r)$ is oscillating

  Then $\psi_0(r) = A\, J_0(k_c r)$

* for $r>a$ (cladding)

  need $k_c^2 < 0$ $\psi_0(r)$ is exp damped

  $\psi_0(r) = B\, K_0(\gamma r)$ where $\gamma^2 = -k_c^2$

X-polarized E field:
Look for solutions w/ $E_y = 0$

$$E_x = \begin{cases} A\, J_0(k_c r) & r<a \\ B\, K_0(\gamma r) & r>a \end{cases} \qquad \text{Then}$$

$\frac{\partial}{\partial z}$ brings down a k_z

$$\vec\nabla\times\vec E = \frac{i\omega}{c}\vec B \Rightarrow \begin{cases} B_x^0 = -\frac{k_z c}{\omega}\underbrace{\cancel{E_y^0}}_{0} - i\frac{c}{\omega}\frac{\partial E_z^0}{\partial y} \approx 0 \\[2ex] B_y^0 = \frac{k_z c}{\omega}E_x^0 + \frac{ic}{\omega}\frac{\partial E_z^0}{\partial x} \approx A\frac{k_z c}{\omega}J_0(k_c r) \end{cases}$$

Since E_z is small

Book shows that $E_z^0 = \frac{ic}{n^2\omega}\left(\frac{\partial B_y^0}{\partial x} - \frac{\partial B_x^0}{\partial y}\right) \approx i\frac{AC}{n^2\omega}\frac{k_z c}{\omega}J_1(k_c r)$

$k_c\cos\phi$

Trick is to define $\frac{\partial}{\partial x}$ in terms of $\frac{\partial}{\partial r}$ & $\frac{\partial}{\partial \phi}$

≈ 0

$$B_z^0 = -\frac{ic}{\omega}\left(\frac{\partial E_y^0}{\partial x} - \frac{\partial E_x^0}{\partial y}\right) \approx iA\sin\phi\,\frac{k_c c}{\omega}J_1(k_c r)$$

B.C. at $r=a$ determine A, B, k_z etc.

$D_\perp$ continuous $\to$ $E_\perp$ continuous since $n_1\approx n_2$

~~E∥ continuous~~ $B_\perp$ continuous

$$\boxed{A J_0(k_c a) = B K_0(\gamma a)}$$

$E_\parallel$ continuous & $H_\parallel$ or $B_\parallel$ continuous

$$\boxed{A k_c J_1(ka) = B\gamma K_1(\gamma a)}$$

$B = \frac{J_0(k_c a)}{K_0(\gamma a)}A$ fixes amplitudes

$$k_c a\frac{J_1(k_c a)}{J_0(k_c a)} = \gamma a\frac{K_1(\gamma a)}{K_0(\gamma a)}$$

fixes k_c, γ & hence k_z

since k_c & γ are functions of k_z

$$k_c^2 = \left(\frac{n_1\omega}{c}\right)^2 - k_z^2$$

$$\gamma^2 = k_z^2 - \left(\frac{n_2\omega}{c}\right)^2$$

$$\Rightarrow k_c^2 + \gamma^2 = \left(\frac{\omega}{c}\right)^2(n_1^2 - n_2^2)$$

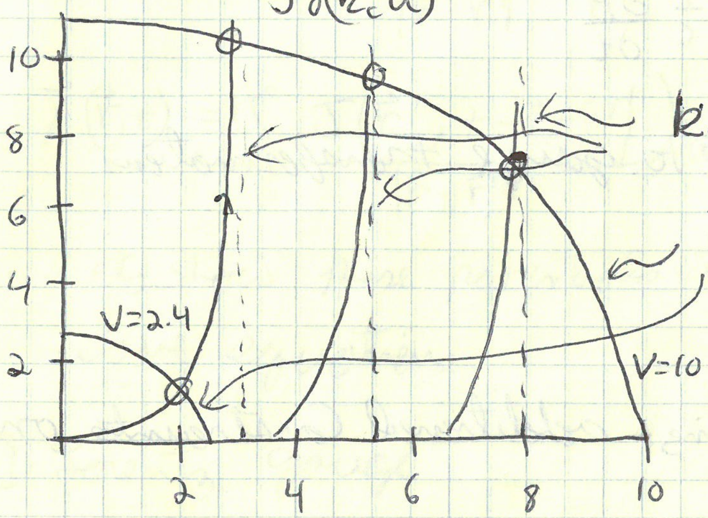
$\to$ Define $V^2 = a^2(k_c^2+\gamma^2)$

$$= \left(\frac{\omega a}{c}\right)^2(n_1^2 - n_2^2)$$

V is a dimensionless parameter that tells about the fiber properties as a function of ω

$$\gamma = \sqrt{\frac{V^2}{a^2} - k_c^2}$$

Plot $\frac{k_c a J_1(k_c a)}{J_0(k_c a)}$ and $\gamma a \frac{K_1(\gamma a)}{K_0 \gamma a}$ vs. $k_c a$



each intersection is a solution labled by a mode $m = 1, 2, \ldots$

mth intersection has $k_c a$ between the mth zero of J_1 & the nth zero of J_0

$$k_{z,m} = \sqrt{\left(\frac{n\omega}{c}\right)^2 - k_{c,m}^2}$$

Such modes are called $\underbrace{L}_{\text{Linearly Polarized}} P_{\ell, m}$ w/ $\ell = 0$ (E, B indep of ϕ)

There are also modes w/ ϕ dependence that have $\ell \neq 0$

# of modes $\approx \frac{V^2}{2}$. No cut off freq. Can find a mode for arbitrary small V.

Single mode fibers:

V small enough that only one mode can propagate

V must be smaller than 3.832 (1st zero of J_1) to avoid LP_{02} mode. LP_{11} mode shows up at $V = 2.405$ (1st zero of J_0). $\therefore$ V must be smaller than 2.405 for a single mode fiber

Caveat: if V is very small then $k_c a \stackrel{\uparrow}{\approx} V$ $(\gamma a)^2 = V^2 - k_c^2 a^2 \downarrow 0$

& Cladding has to be very thick for evanescent wave to decay completely