New topic Retarded potentials

a) Recall: $\begin{cases} \vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \\ \vec{B} = \vec{\nabla}\times\vec{A} \end{cases}$

$\vec{A}, \Phi$ determined up to gauge transformations

$$\vec{A} \to \vec{A} + \vec{\nabla}\xi$$

$$\Phi \to \Phi - \frac{1}{c}\frac{\partial \xi}{\partial t}$$

Fix gauge by imposing additional constraints on $\vec{A}, \Phi$:

"Lorentz gauge" $\vec{\nabla}\cdot\vec{A} = -\frac{1}{c}\frac{\partial \Phi}{\partial t}$

Then $$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -4\pi\rho$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J}$$

Two limits:

Quasi static: Neglect 2nd order time derivatives

$$\Phi(\vec{r}) = \int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,dv'$$

$$\vec{A}(\vec{r}) = \frac{1}{c}\int_V \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,dv'$$

Rapidly changing fields:

Since changes in charge configurations affect fields & since these effects can only propagate at the speed of light, we need to introduce the retarded time. $t - \frac{|\vec{r}-\vec{r}\,'|}{c}$ so that we can write the fields as a function of retarded time rather than assuming they respond instantly

$$\Phi(\vec{r},t) = \int_V \frac{\rho\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} dV'$$

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int \frac{J\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} dV'$$

Lets show these potentials satisfy the Lorentz gauge & wave equations

① Lorentz gauge

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{c}\int_V \vec{\nabla}_r \cdot \frac{J\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} dV' \qquad \text{use } \vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$$

$$= \frac{1}{c}\int_V -\frac{\partial}{\partial t} \frac{\rho\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} dV' = \frac{1}{c}\frac{\partial}{\partial t}\Phi$$

② wave eq:

Use the following math results:

- $\nabla^2 \frac{1}{|\vec{r}-\vec{r}'|} = -4\pi\delta(\vec{r}-\vec{r}')$ Green's function

- $\vec{\nabla}(\vec{r}-\vec{r}') = \hat{e}_{\vec{r}-\vec{r}'}$

  $\vec{\nabla}\cdot\hat{e}_{\vec{r}-\vec{r}'} = \frac{2}{|\vec{r}-\vec{r}'|}$

- $\nabla^2|\vec{r}-\vec{r}'| = \frac{2}{|\vec{r}-\vec{r}'|}$

- $\nabla\frac{1}{|\vec{r}-\vec{r}'|} = -\frac{1}{|\vec{r}-\vec{r}'|^2}\hat{e}_{\vec{r}-\vec{r}'}$

$$\vec{\nabla}_r\Phi = \int_V dV'\left\{\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|}\rho - \frac{1}{c}\frac{1}{|\vec{r}-\vec{r}'|}\underbrace{\frac{\partial\rho}{\partial t'}}_{\frac{\partial\rho}{\partial t}}\nabla(\vec{r}-\vec{r}')\right\}$$

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi = -\frac{1}{c^2}\int_V dV'\frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial^2}{\partial t^2}\rho$$

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi =$$

$$\int_V dv' \, \rho\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right) \nabla^2 \frac{1}{|\vec{r}-\vec{r}'|} - \frac{2}{c}\int_V dv' \frac{\partial \rho}{\partial t}\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right) \vec{\nabla}|\vec{r}-\vec{r}'| \cdot \nabla \frac{1}{|\vec{r}-\vec{r}'|}$$

$$- \frac{1}{c}\int_V dv' \frac{\partial \rho}{\partial t}\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right) \frac{1}{|\vec{r}-\vec{r}'|} \nabla^2 |\vec{r}-\vec{r}'| + \frac{1}{c^2}\int_V dv' \frac{\partial^2 \rho}{\partial t^2}\left(\vec{r}', t - \frac{|r-r'|}{c}\right) \frac{1}{|r-r'|}\left((\nabla|r-r'|)^2 - 1\right)$$

$$= -4\pi \int_V dv' \, \rho\left(r', t - \frac{|\vec{r}-\vec{r}'|}{c}\right) \delta(r-r') = -4\pi \rho(\vec{r}, t)$$

Proof for $\vec{A}$ goes similarly

notation: $\rho\left(r', t - \frac{|\vec{r}-\vec{r}'|}{c}\right) = [\rho(\vec{r}')]$

$\vec{J}\left(r', t - \frac{|\vec{r}-\vec{r}'|}{c}\right) = [J(\vec{r}')]$

d) Retarded Fields

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

as before

$$\vec{E} = -\int_V dv' \, \nabla\left\{\frac{[\rho(\vec{r}')]}{|\vec{r}-\vec{r}'|}\right\} - \frac{1}{c^2}\int_V dv' \frac{d}{dt}\frac{[J(\vec{r}')]}{|\vec{r}-\vec{r}'|} =$$

$$+ \int_V dv' \left\{ \frac{\rho\left(\vec{r}, t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|^2} \hat{e}_{\vec{r}-\vec{r}'} \right.$$

$$+ \frac{1}{c}\frac{\frac{d}{dt}\rho\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} \hat{e}_{\vec{r}-\vec{r}'}$$

$$\left. - \frac{1}{c^2}\frac{\frac{d}{dt}\vec{J}\left(r', t - \frac{|r-r'|}{c}\right)}{|r-r'|} \right\}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\nabla\times\left(\frac{\vec{J}\left(r, t-\frac{|\vec{r}-\vec{r}'|}{c}\right)}{|r-r'|}\right) = -\frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial\vec{J}}{\partial t}\left(\vec{r}', t-\frac{|\vec{r}-\vec{r}'|}{c}\right)\times\vec{\nabla}\left(-\frac{|\vec{r}-\vec{r}'|}{c}\right)$$
$$-\vec{J}\left(\vec{r}', t-\frac{|\vec{r}-\vec{r}'|}{c}\right)\times\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{c}\frac{\partial\vec{J}}{\partial t}\left(\vec{r}', t-\frac{|\vec{r}-\vec{r}'|}{c}\right)\times\frac{\hat{e}_{\vec{r}-\vec{r}'}}{|\vec{r}-\vec{r}'|} + \vec{J}\left(\vec{r}', t-\frac{|\vec{r}-\vec{r}'|}{c}\right)\times\frac{\hat{e}_{\vec{r}-\vec{r}'}}{|\vec{r}-\vec{r}'|^2}.$$

$$\vec{B}(\vec{r},t) = \int_V dv'\left\{\frac{\vec{J}\left(\vec{r}', t-\frac{|\vec{r}-\vec{r}'|}{c}\right)\times\hat{e}_{\vec{r}-\vec{r}'}}{c\,|\vec{r}-\vec{r}'|^2} + \frac{\frac{\partial\vec{J}}{\partial t}(\cdots)\times\hat{e}_{\vec{r}-\vec{r}'}}{c^2\,|\vec{r}-\vec{r}'|}\right.$$

retarded Biot savart law ~~[illegible]~~

notice that:

$$\vec{J}\left(\vec{r}, t-\frac{|\vec{r}-\vec{r}'|}{c}\right) = \vec{J}(\vec{r},t) - \frac{|\vec{r}-\vec{r}'|}{c}\frac{\partial}{\partial t}\vec{J}(\vec{r},t) + \frac{1}{2}\left(\frac{|\vec{r}-\vec{r}'|}{c}\right)^2\frac{\partial^2}{\partial t^2}\vec{J}(\vec{r},t)+\ldots$$

$$B(\vec{r},t) = \int_V \frac{\vec{J}(\vec{r}',t)\times\hat{e}_{\vec{r}-\vec{r}'}}{c\,|\vec{r}-\vec{r}'|^2}dv' + \text{terms involving second derivative of } \vec{J}$$

Similarly

$$\rho\left(r, t-\frac{|\vec{r}-\vec{r}'|}{c}\right) = \rho(\vec{r},t) - \frac{|r-r'|}{c}\frac{\partial}{\partial t}\rho(\vec{r},t) + \frac{1}{2}\left(\frac{|\vec{r}-\vec{r}'|}{c}\right)^2\frac{\partial^2}{\partial t^2}\rho(\vec{r},t)+\ldots$$

$$\vec{E} = \int_V \frac{\rho(\vec{r}'t)\,\hat{e}_{\vec{r}-\vec{r}'}}{|\vec{r}-\vec{r}'|^2} - \frac{1}{c}\frac{\partial\vec{J}/\partial t\,(r,t)}{|\vec{r}-\vec{r}'|^2} + \text{terms that involve second derivatives of } \rho, \vec{J}$$

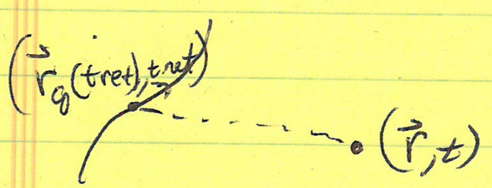
The new contributions to $\vec{E}$ & $\vec{B}$ decay as $\frac{1}{|\vec{r}-\vec{r}'|}$ <u>not</u> $\frac{1}{|\vec{r}-\vec{r}'|^2}$ if $|\vec{r}-\vec{r}'|$ is large

They dominate at large distances. Physically they correspond to <u>radiation</u>

Lienard Wiechert Potentials

potentials & fields of a moving charge.

define $\hat{n} = \frac{\vec{R}}{|R|}$, $\vec{\beta} = \frac{\vec{u}}{c}$, $K = 1 - \vec{\beta}\cdot\vec{n}$



$$\Phi = \frac{q}{R - \vec{\beta}\cdot\vec{R}}$$

$$\vec{A} = \frac{q\vec{\beta}}{R - \vec{\beta}\cdot\vec{R}}$$

$$\vec{R} = \vec{r} - \vec{r}_q(t_{ret})$$

$$\beta = \frac{1}{c}\vec{u}_q(t_{ret}) = \frac{1}{c}\frac{\partial \vec{r}_q(t_{ret})}{\partial t_{ret}}$$

$$t_{ret} = t - \frac{|\vec{r} - \vec{r}_q(t_{ret})|}{c}$$

$$E(\vec{r},t) = q\left[\frac{(\vec{R} - \vec{\beta}R)(1-\beta^2)}{(R - \vec{\beta}\cdot\vec{R})^3} + \frac{(\vec{R} \times ((\vec{R} - \vec{\beta}R)\times a)}{c^2(R - \vec{\beta}\cdot\vec{R})^3}\right]$$

$$B(r,t) = q\left[\frac{(\vec{\beta}\times\vec{R})(1-\beta^2)}{(R - \vec{\beta}\cdot\vec{R})^3} + \frac{(\vec{a}\cdot\vec{R})(\vec{\beta}\times\vec{R})}{c^2(R - \vec{\beta}\cdot\vec{R})^3} + \frac{\vec{a}\times\vec{R}}{c^2(R - \beta\cdot R)}\right]$$

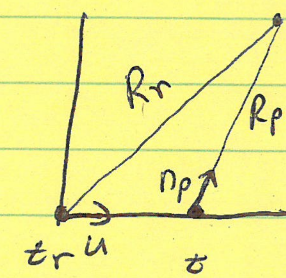
$$= \frac{\vec{R}\times\vec{E}}{R}$$

1) without accelerated charges $E, B \propto \frac{1}{R^2}$
with accel. charges $E, B \propto \frac{1}{R}$
Energy flux $\propto EB$
Energy flux through sphere $\propto EBR^2$
finite regardless of how far away we are from moving charge — this is radiation

Some results

2) Charged particle at constant velocity $\vec{u}$

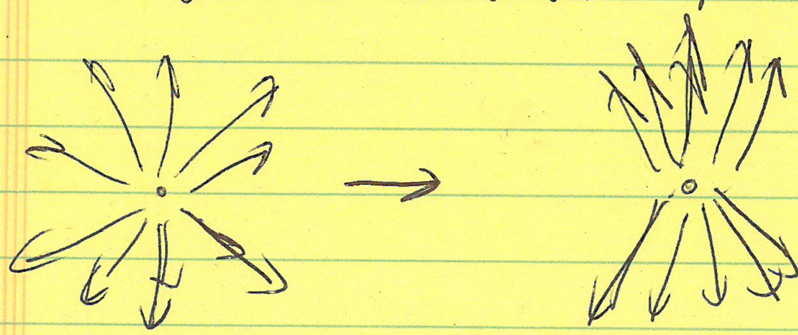
want to express E in terms of the vector ~~$\vec{R}$~~ to the field point from the present position R_p of the particle instead of vector from retarded position R_r



$$\vec{E} = \frac{e\,(1-\beta^2)}{R_p^2\,(1-\beta^2\sin^2\theta_p)^{3/2}}\,\hat{n}_p$$

$$\vec{B} = \beta \times E = \frac{e\,(1-\beta^2)}{R_p^2\,(1-\beta^2\sin^2\theta_p)^{3/2}}\,\vec{\beta}\times\hat{n}_p$$

E field is radial from present position even though field originates at retarded position

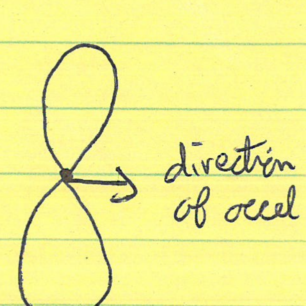


3) accelerated particle low velocity $\beta \ll 1$ $K \approx 1$

$$\vec{E}_a = \frac{q}{c^2 R}\left(\hat{n}\times(\hat{n}\times\vec{a})\right) \qquad \vec{B}_a = \hat{n}\times\vec{E}_a$$

$$= \frac{q}{c^2 R}\left(\hat{n}(\hat{n}\cdot\vec{a}) - \vec{a}\right)$$

$$E_a^2 = \frac{q^2}{c^4 R^2}\left(a^2 - (\hat{n}\cdot\vec{a})^2\right) = \frac{q^2 a^2}{c^4 R^2}\sin^2\theta$$



$$S = \frac{c}{4\pi}\left(\vec{E}_a \times \vec{B}_a\right) = \frac{c}{4\pi} E_a^2\,\hat{n} = \frac{q^2 a^2 \sin^2\theta}{4\pi c^3 R^2}\,\hat{n}$$

Power radiated per solid angle $d\Omega$

$$\frac{dP}{d\Omega} = \vec{S}_a \cdot \hat{n} R^2 = \frac{q^2 a^2 \sin^2\theta}{4\pi c^3}$$

total power radiated

$$P = \int d\Omega \, \frac{dP}{d\Omega} = \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \, \frac{dP}{d\Omega}$$

$$= \frac{2q^2a^2}{3c^2}$$

Larmor formula

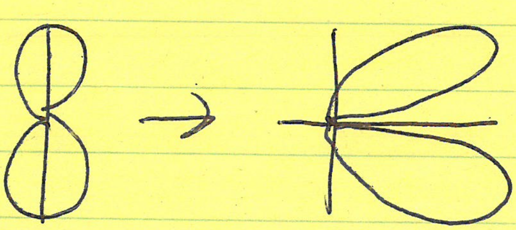
4) accelerated particle velocity comparable to c

$\vec{a}$ & $\vec{u}$ colinear

$$\vec{S}_a = \frac{q^2 a^2 \sin^2\theta}{c^4 K^6 R^2}\,\hat{n}$$

$$\frac{dP}{d\Omega} = \frac{q^2 a^2 \sin^2\theta}{4\pi c^3 (1-\beta\cos\theta)^5}$$

θ: angle between velocity and retarded position vector $\vec{R}$



as $\beta \to 1$ $\frac{dP}{d\Omega}$ is strongly peaked in forward direction

$$P = \int d\Omega \, \frac{dP}{d\Omega} = \frac{2q^2a^2}{3c^2}\,\frac{1}{(1-\beta^2)^3}$$

useful for Bremsstrahlung radiation emitted by accelerating electrons

Antennas

1) Lect 29 of 2009 notes

2) Electric dipole radiation

Describe time-dependent $\rho, \vec{J}$ in terms of microscopic charges. We will assume we are in the non-relativistic Limit $\vec{\beta} \equiv \frac{u}{c} << 1$

$\hat{n}_\alpha$ & $\hat{r}_\alpha$ point from charge to field point

$$\vec{E}_{rad} = \sum_\alpha \frac{q_\alpha}{c^2 r_\alpha}\left(\hat{n}_\alpha \times (\hat{n}_\alpha \times \vec{a}_\alpha)\right)$$

since $r - r_\alpha$ is small as $\vec{r} \to \infty$

$$\vec{E}_{rad} \cong \sum_\alpha \frac{q_\alpha}{c^2 r}\left(\hat{n} \times (\hat{n} \times \vec{a}_\alpha)\right)$$

& since $\hat{n} - \hat{n}_\alpha$ is small as $\vec{r} \to \infty$

$$= \frac{1}{c^2 r}\left(\hat{n} \times \Big(\hat{n} \times \underbrace{\Big(\sum_\alpha q_\alpha \vec{a}_\alpha\Big)}_{\ddot{\vec{p}}}\Big)\right)$$

$$= \frac{1}{c^2 r}\left(\hat{n} \times (\hat{n} \times \ddot{\vec{p}})\right)$$

$\ddot{\vec{p}} \leftarrow$ dipole moment

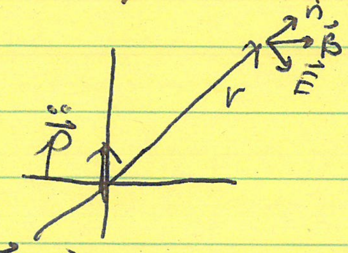
$$\vec{B}_{rad} = \hat{n} \times \vec{E}_{rad} = \frac{-1}{c^2 r}\, \hat{n} \times \ddot{\vec{p}}$$

with $\ddot{\vec{p}}$ evaluated at the retarded time

In polar coordinates with $\ddot{\vec{p}}$ along z axis $\ddot{\vec{p}} = \ddot{p}\, e_z$

$$\vec{E}_{rad} = \frac{\ddot{p}}{c^2 r} \sin\theta\, \hat{e}_\theta$$

$$\vec{B}_{rad} = \frac{\ddot{p}}{c^2 r} \sin\theta\, \hat{e}_\phi$$



$\vec{E}, \vec{B}, \hat{n}$ form right handed set of axes.

Same field as a particle w/ acceleration $\vec{a} = \ddot{\vec{p}}/q$ and charge q

recall power radiated for one charge q

$$S = \frac{c}{4\pi}(E_{rad} \times B_{rad}) = \frac{q^2 a^2 \sin^2\theta}{4\pi c^3 R^2}\, \hat{n} \qquad \frac{dP}{d\Omega} = (\vec{S}_{rad} \cdot \hat{n}) R^2 = \frac{q^2 a^2 \sin^2\theta}{4\pi c^3}$$

for many charges we just sum up the q's.

$$\frac{dP}{d\Omega} = \frac{(\ddot{p})^2 \sin^2\theta}{4\pi c^3} \qquad P = \frac{2(\ddot{p})^2}{3c^3} \equiv \text{total power radiated}$$

Ex: Harmonic time dep

if $p(t) = p_0 e^{-i\omega t}$

$$\ddot{p}^2 = p_0^2 \omega^4 \cos^2\omega t \qquad \left\langle \frac{dP}{d\Omega} \right\rangle = \frac{p_0^2 \omega^4 \sin^2\theta}{8\pi c^3}$$

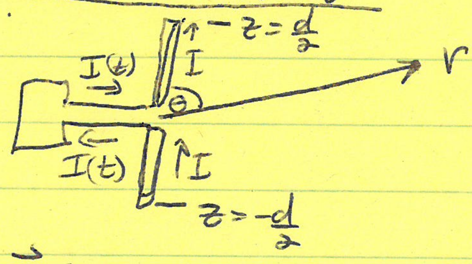
$$\langle \ddot{p}^2 \rangle = \frac{1}{2} p_0^2 \omega^4 \qquad \langle P \rangle = \frac{p_0^2 \omega^4}{3c^3}$$

this is only ~~s~~ is B ...

Usually this is much smaller than ohmic losses in the conductor. Ohmic losses make these antennas impractical

3) More practical antennas ~~are~~ have dimensions comparable to λ so dipole ~~approx~~ fails.

Linear antennas



Center driven antenna

$$\vec{J}(\vec{r}', t_r')\, dv' \longrightarrow I(z', t_r')\, dz'\, \hat{e}_z$$

precise current pattern in antenna is difficult to calculate. assume sinusoidal dependence with nodes at $z = \pm \frac{d}{2}$

$$I(z', t) = I_0 e^{-i\omega t_r} \sin k\left(\frac{d}{2} - |z|\right)$$

where $k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$

$I(z, t_r) = I(-z, t_r)$ due to device symmetry

$$I(0, t) = I_0 e^{-i\omega t_r} \sin\frac{kd}{2}$$

Calculate radiation fields assuming $r \gg \lambda$
so that radiation fields dominate

$$\vec{B}_{rad} = \int_V dV' \frac{\frac{\partial \vec{J}}{\partial t}(r', t'_r) \times \hat{e}_{r-r'}}{c^2 |r-r'|} = -i\omega \frac{\hat{e}_z \times \hat{e}_{\vec{r}}}{c^2 r} \int_{-d/2}^{d/2} dz' I(z', t(z'))$$

$$\vec{E}_{rad} = -\hat{e}_{\vec{r}} \times \vec{B}_{rad}$$

$$\vec{S}_{rad} = \frac{c}{4\pi} \vec{E}_{rad} \times \vec{B}_{rad} = \frac{c}{4\pi} B_{rad}^2 \hat{e}_r$$

So it all boils down to integral $\int_{-d/2}^{d/2} dz' I(z', t(z'))$

expand around $t'(0)$

$$t'(z') = t - \frac{|\vec{r} - z'\hat{e}_z|}{c} = t - \frac{r}{c} + \frac{z'}{c}\cos\theta - \frac{1}{2}\frac{z'^2}{cr}\sin^2\theta + \ldots$$

where we used
$$|\vec{r} - z'\hat{e}_z| = \sqrt{r^2 - 2z'r\cos\theta + z'^2}$$
$$= r\sqrt{1 - \frac{2z'}{r}\cos\theta + \frac{z'^2}{r^2}}$$
$$\approx r\left(1 - \frac{z'}{r}\cos\theta - \frac{1}{2}\frac{z'^2}{r^2}(\cos^2\theta - 1) + \ldots\right)$$
$$= r - z'\cos\theta + \frac{1}{2}\frac{z'^2}{r}\sin^2\theta + \ldots$$

$$I(z', t'(z')) = I_0 e^{-i\omega t'} \sin k\left(\frac{d}{2} - |z'|\right)$$
$$= I_0 e^{-i\omega t + i\omega r/c} e^{-i\frac{\omega}{c} z'\cos\theta + \frac{1}{2} i\frac{\omega}{c}\frac{z'^2}{r}\sin^2\theta} \sin k\left(\frac{d}{2} - |z'|\right)$$

For integration range of interest $\frac{\omega}{c} z'$ term varies appreciably. Since $d \sim \lambda/2$ in this case.

whether $\frac{\omega}{2c}\frac{z'^2}{r}$ is important depends on r

If $r \gg d^2 \frac{\omega}{c} \sim \frac{d^2}{\lambda}$ this term can be neglected

"Fraunhofer Limit" $r \gg \frac{d^2}{\lambda}$

all three limits can be combined as:

$$d \ll \sqrt{\lambda r} \ll r$$

now we can integrate I

$$\int_{-d/2}^{d/2} I(z', t(z'))\,dz' \approx \int_{-d/2}^{d/2} I_0 e^{-i\omega t + i\omega \frac{r}{c}} e^{-ikz'\cos\theta} \sin k\left(\frac{d}{2} - |z'|\right)$$

$$= 2I_0\, e^{-i\omega t + i\omega \frac{r}{c}} \frac{\cos\left(\frac{dk\cos\theta}{2}\right) - \cos\frac{dk}{2}}{k\sin^2\theta}$$

$$\vec{B}_{rad} = \frac{-2i\omega I_0 \sin\theta\, \hat{e}_\phi}{rc^2 k \sin^2\theta} e^{-i\omega t + i\omega r/c} \left\{\cos\frac{kd\cos\theta}{2} - \cos\frac{kd}{2}\right\}$$

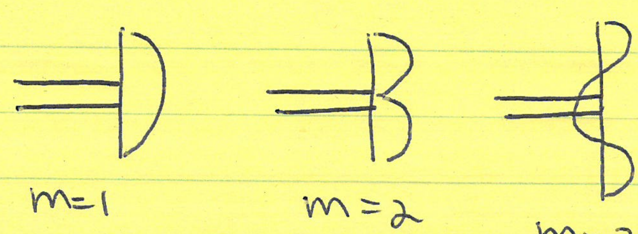
$$= \frac{2iI_0}{rc} e^{i(kr-\omega t)} \frac{\cos\left(\frac{kd\cos\theta}{2}\right) - \cos\frac{kd}{2}}{\sin\theta}$$

$$\left(\frac{dP}{d\Omega}\right) = r^2 \langle S_{rad}\rangle \cdot \hat{e}_{\vec{r}} = \frac{cr^2}{4\pi}\langle B_{rad}^2\rangle$$

$$= \frac{I_0^2}{2\pi c}\left(\frac{\cos\left(\frac{kd\cos\theta}{2}\right) - \cos\frac{kd}{2}}{\sin\theta}\right)^2$$

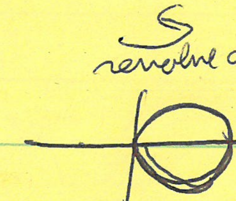
angular distribution depends on $\frac{kd}{2}$

define $m = \frac{d}{(\lambda/2)} = \frac{kd}{\pi}$ counts # of half wavelengths in the antenna



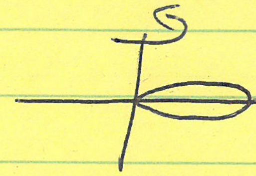
m=1 m=2 m=3

$$\left\langle \frac{dP}{d\Omega} \right\rangle_{m=1} = \frac{I_0^2}{2\pi c} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$



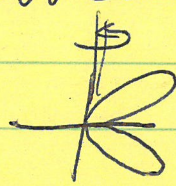
m=1

$$\left\langle \frac{dP}{d\Omega} \right\rangle_{m=2} = \frac{I_0}{2\pi c} \frac{4\cos^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$



m=2

off center feed can give different patterns



m=2

antenna arrays add & subtract fields to shape your signal