Lect 31) Relativistic Electrodynamics

* Up to now all theory was relativistically correct. When we had results that were for $v/c \ll 1$ only, that was because we made a mathematical approximation of a theory that was valid for all v

* Why then do we need to spend time talking about special relativity?

① Because mechanics is different → we want to know how charged particles move under influence of EM fields

② It teaches us how to relate $\vec{E}$ & $\vec{B}$ fields in different frames of reference.

③ Elegant formulation of certain hard won results of standard EM theory.

1) Basic Postulates

a) all physical laws are the same in all inertial systems

b) Velocity of light (in free space) is a universal constant indep of the motion of the source (follows from a)

2) Classical mechanics defined by Newton's Eqs of motion

$F_j = m\ddot{x}_j \quad j = 1,2,3$ is invariant with respect to Galilean transformations

$$\begin{cases} t' = t \\ x'_j = x_j - V_j t \quad j = 1,2,3 \end{cases}$$

other invariants are the duration of a time interval

$$dt = dt'$$

and a distance (taken at same time)

$$ds^2 = \sum_j dx_j^2 = ds'^2$$

But: Maxwell's eqs, in particular the wave eq for EM radiation

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = 0 \qquad \psi = E_j, B_j$$

is not invariant. It transforms to

$$\nabla'^2 \psi - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\psi - \frac{1}{c^2}(\vec{V}\cdot\vec{\nabla})^2\psi + \frac{2}{c}(\vec{V}\cdot\vec{\nabla})\frac{\partial}{\partial t'}\psi$$

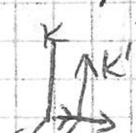
$\therefore$ If Maxwell's eq's form the correct theory for EM we need to find new transformations between inertial reference frames.

also Newton's eq of motion cannot be the correct theory since it is invariant under Galilean transformation

Who's right? Experiments say Maxwell

3) Lorentz transformation:

If the wave eq is invariant, then a light pulse emitted at the common origin of two reference frames K, K' must travel equally fast.



$$\sum_{j=1}^{3} x_j^2 - c^2 t^2 = 0$$ is equation for light pulse in K

$$\sum_{j=1}^{3} x_j'^2 - c^2 t'^2 = 0$$ eq in K'

define $X_\mu \begin{cases} x_j & j = \mu = 1,2,3 \\ ict & \mu = 4 \end{cases}$ $\qquad \sum_\mu X_\mu^2 = X_\mu X_\mu = 0$

Summation Convention

$X'_\mu X'_\mu = 0$

* The only linear transformation

$$x_\mu \to x'_\mu = \sum_\nu \lambda_{\mu\nu} x_\nu = \lambda_{\mu\nu} x_\nu \quad \text{implicit summation}$$

that obeys this property is (for $\vec{v}$ along 3 axis):

$$\lambda = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \gamma & i\beta\gamma \\ & & -i\beta\gamma & \gamma \end{pmatrix} \qquad \text{where } \gamma = \frac{1}{\sqrt{1-\beta^2}}, \ \beta = \frac{v}{c}$$

or

$$x'_1 = x_1$$
$$x'_2 = x_2$$
$$x'_3 = \gamma(x_3 - vt)$$
$$t' = \gamma\left(t - \frac{\beta}{c} x_3\right)$$

or

$$\vec{x}' = \vec{x} + \vec{v}\left(\frac{\vec{x}\cdot\vec{v}}{v^2}(\gamma-1) - \gamma t\right)$$
$$t' = \gamma\left(t - \frac{\vec{x}\cdot\vec{v}}{c^2}\right)$$

The norm of the 4 dim position/time vector is invariant under Lorentz transformations

$$x'_\mu x'_\mu = x_\nu \lambda_{\mu\nu} \lambda_{\mu\rho} x_\rho = x_\nu \underbrace{(\lambda^T \lambda)_{\nu\rho}}_{\substack{\mathbb{1} \text{ identity matrix} \\ \delta_{\nu\rho}}} x_\rho = x_\nu x_\nu$$

Any 4 dim vector that transforms according to the Lorentz transformation is called a "Four Vector"

ex: $\underline{x} = (x_1\, x_2\, x_3, \underbrace{ict}_{x_4}) = (\vec{x}, ict)$

an invariant is ~~an~~ a scalar object that doesn't change in a Lorentz transformation

ex: $x_\mu x_\mu$ or $ds^2 = dx_\mu dx_\mu$

notation: $d\tau = \frac{i}{c} ds = \frac{i}{c}\sqrt{dx_\mu dx_\mu} = \sqrt{dt^2 - \frac{1}{c^2} dx_j dx_j}$

is called "proper time" both ds & $d\tau$ are invariants

- other 4 vectors

$$\underline{U} = \frac{dx}{d\tau} \text{ or } U_\mu = \frac{dx_\mu}{d\tau} \quad \text{four } \cancel{\text{scalar}} \text{ vector velocity}$$

$$= \left(\frac{d\vec{x}}{d\tau}, ic\frac{dt}{d\tau}\right)$$

in terms of the ordinary velocity $u_j = \frac{dx_j}{dt}$ one has

$$d\tau = \sqrt{dt^2 - \frac{1}{c^2} dx_j dx_j}$$

$$= dt\left(1 - \frac{1}{c^2}\frac{dx_j}{dt}\frac{dx_j}{dt}\right)^{1/2} = dt\sqrt{1-\beta^2}$$

$$\underline{U} = \left(\frac{\vec{u}}{\sqrt{1-\beta^2}}, \frac{ic}{\sqrt{1-\beta^2}}\right)$$

- a trivial invariant is m_0 a particle's mass in its rest frame

$m_0\underline{U}$ is also a four vector $\Rightarrow$ momentum

$$\underline{P} = m_0\underline{U} = \left(\frac{m_0 \underline{\underline{u}}}{\sqrt{1-\beta^2}}, \frac{i m_0 c}{\sqrt{1-\beta^2}}\right)$$

~~order~~ interpretation of fourth component:

"Force": $F = \frac{d}{dt}\vec{p} = \frac{d}{dt}\left(\frac{m_0 \underline{\underline{u}}}{\sqrt{1-\beta^2}}\right)$

Time derivative of K.E. is just the work done on particle per unit time

$$\frac{dT}{dt} = \vec{F}\cdot\vec{u} = \vec{u}\cdot\frac{d}{dt}\left(\frac{m_0\vec{u}}{\sqrt{1-\beta^2}}\right) = m_0c^2\frac{d}{dt}\left(\frac{1}{\sqrt{1-\beta^2}}\right),$$

not trivial
need to work
this through

$$\int_{t_1}^{t_2}\frac{dT}{dt} = T_2 - T_1 = \frac{m_0c^2}{\sqrt{1-\beta_2^2}} - \frac{m_0c^2}{\sqrt{1-\beta_1^2}}$$

is object ~~is~~ at rest at t_1 $\quad T(t_2) = \frac{m_0c^2}{\sqrt{1-\beta^2}} - \underbrace{m_0c^2}_{\text{rest energy}}$

total energy $W = T + m_0c^2 = \frac{m_0c^2}{\sqrt{1-\beta^2}}$

$\beta \ll 1 : \quad T \approx \frac{1}{2}m_0u^2$

$$P_4 = \frac{im_0c}{\sqrt{1-\beta^2}} = i\frac{W}{c} \qquad \underline{P} = \left(\vec{P}, i\frac{W}{c}\right)$$

Length Contraction:

Look at volume V described by

$0 < x_1 < dx_1$ in mom's frame

$0 < x_2 < dx_2$

$0 < x_3 < dx_3$

$$x_3 = \gamma x_3' + \gamma v t'$$

hence $x_3 = 0$ is mapped to

$$\gamma x_3' + \gamma v t' = 0 \iff x_3' = -vt'$$

$x_3 = dx_3$ is mapped to: $\quad dx_3 = \gamma x_3' + \gamma v t'$

$$\iff x_3' = -vt' + \frac{1}{\gamma}dx_3$$

$$dx_3' = dx_3\sqrt{1-\beta^2}$$

Since ds is invariant length contraction implies time dilation

5) taking Derivatives

$$\underline{\nabla} = \hat{e}_\mu \frac{\partial}{\partial x_\mu} = \sum_{j=1}^{3} \hat{e}_j \frac{\partial}{\partial x_j} + \frac{1}{ic} \hat{e}_4 \frac{\partial}{\partial t}$$ "four gradient"

Transformation rule:

$$\frac{\partial}{\partial x'_\mu} = \underbrace{\frac{\partial x_\nu}{\partial x'_\mu}} \frac{\partial}{\partial x_\nu}$$

$$x'_\mu = \lambda_{\mu\nu} x_\nu \Leftrightarrow x_\nu = \left(\lambda^T\right)_{\nu\mu} x'_\mu$$

$$\frac{\partial x_\nu}{\partial x'_\mu} = \lambda_{\mu\nu} \quad \& \quad \frac{\partial}{\partial x'_\mu} = \lambda_{\mu\nu} \frac{\partial}{\partial x_\nu}$$

$\underline{\nabla}^2 = \frac{\partial^2}{\partial x_\mu^2} \equiv \Box^2$ is invariant

wave eq $\Box^2 \psi = 0$ remains unchanged under Lorentz transform

6) Charge & Currents

$\underline{J} = \left(\vec{J}, ic\rho\right)$ is a four vector

Proof: first look at transformation law for ρ

Total charge q must be an invariant but ρ, the charge density need not! If charge density in rest frame is ρ_0, then charge in volume dV is $dq = \rho_0 \, dV$

In moving frame K', $dq' = dq$ (assume K' moves at velocity $v\hat{e}_3$ w.r.t rest frame)

$$dV' = dx_1' dx_2' dx_3'$$
$$= dx_1 \, dx_2 \, dx_3 \left(\sqrt{1-\beta^2}\right)$$

$$\rho' = \frac{dq'}{dV'} = \frac{1}{\sqrt{1-\beta^2}} \frac{dq}{dV} = \gamma \rho_0$$

Hence $\rho = \dfrac{\rho_0}{\sqrt{1-\beta^2}}$

Current density in K'

$$\vec{J} = \rho\vec{u} = \frac{\rho_0 \vec{u}}{\sqrt{1-\beta^2}}$$

Hence, in K'

$\underline{J} = \rho_0 \underline{u}$ Since $\underline{u}$ is a 4-vector, so is $\underline{J}$

7) Potentials A, Φ

a) Gauge: Lorentz Gauge

$$\mathrm{div} A + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0$$

write $\underline{A} = (\vec{A}, i\Phi)$ then Lorentz gauge is: $\underline{\nabla} \cdot \underline{A} = \frac{\partial}{\partial x_\mu} A_\mu = 0$

b) Equations for A, Φ:

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c} J$$

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -4\pi\rho$$

rewrite as $\Box^2 \underline{A} = -\frac{4\pi}{c}\underline{J}$

Since $\Box^2$ is invariant, & $\underline{J}$ is a 4-vector, $\underline{A}$ must be a 4-vector

Fields E & B

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

a) Combine $\vec{E}, \vec{B}$ in antisymmetric tensor (6 indep elements)

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix} = -F_{\nu\mu}$$

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}$$

b) Maxwell's Eqs

$$\frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} = 0$$

all indecis are different no summation

Check: if two indecies are equal equation is trivial

If all three are different & none of them is 4

$$\frac{\partial F_{12}}{\partial x_3} + \frac{\partial F_{23}}{\partial x_1} + \frac{\partial F_{31}}{\partial x_2} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

If all indecies are different, & one of them is 4 & the other two indecies do not include j:

$$\left(\vec{\nabla}\times\vec{E}\right)_j + \frac{1}{c}\frac{\partial B_j}{\partial t} = 0 \qquad j=1,2,3 \quad \Rightarrow \quad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

Hence these permutations reproduce the homogeneous maxwell equations. The inhomogeneous eqs. follow from

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{4\pi}{c} J_\mu$$

summation over ν

For $\mu = j$, $j = 1,2,3$ this gives $\left(\vec{\nabla}\times\vec{B}_j\right) - \frac{1}{c}\frac{\partial E_j}{\partial t} = \frac{4\pi}{c} J_j$

For $\mu = 4$ one finds $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$

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c) We have seen that the continuity eq

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0$$

follows from the Maxwell eqs:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \Rightarrow \frac{\partial \rho}{\partial t} = \frac{1}{4\pi}\vec{\nabla}\cdot\frac{\partial \vec{E}}{\partial t}$$

$$= \frac{1}{4\pi}\nabla\cdot\left(-4\pi\vec{J} + \vec{\nabla}\times\vec{B}\right)$$

$$= -\nabla\cdot J$$

with relativistic notation

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0 \quad \text{becomes}$$

$$\frac{\partial}{\partial x_\mu} J_\mu = 0$$

Proof: $$\frac{\partial J_\mu}{\partial x_\mu} = \frac{c}{4\pi}\frac{\partial F_{\nu\mu}}{\partial x_\mu \partial x_\nu}$$

$$= -\frac{c}{4\pi}\frac{\partial F_{\mu\nu}}{\partial x_\mu \partial x_\nu} = -\frac{c}{4\pi}\frac{\partial F_{\mu\nu}}{\partial x_\nu \partial x_\mu}$$

relabel $\nu \to \mu$

$$= -\frac{c}{4\pi}\frac{\partial F_{\nu\mu}}{\partial x_\mu \partial x_\nu}$$

$$\frac{\partial J_\mu}{\partial x_\mu} = -\frac{\partial J_\mu}{\partial x_\mu} = 0$$

d) Transformation Properties

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}$$

Both A_ν & $\frac{\partial}{\partial x_\mu}$ are 4-

$$F'_{\mu\nu} = \frac{\partial A'_\nu}{\partial x'_\mu} - \frac{\partial A'_\mu}{\partial x'_\nu}$$

$$F'_{\mu\nu} = \lambda_{\nu\rho}\lambda_{\mu\sigma}\left(\frac{\partial A_\rho}{\partial X_\sigma} - \frac{\partial A_\sigma}{\partial X_\rho}\right)$$

$$= \lambda_{\nu\rho}\lambda_{\mu\sigma} F_{\rho\sigma}$$

in matrix language:

$$F'_{\mu\nu} = \lambda_{\nu\rho} F_{\rho\sigma} (\lambda^T)_{\sigma\mu}$$

$$F' = \lambda \underline{\underline{F}} \lambda^T$$

Carrying out the matrix multiplication

$$E_1' = \gamma(E_1 - \beta B_2), \quad E_2' = \gamma(E_2 + \beta B_1) \quad \Big\} \quad \vec{E}_\perp' = \gamma\left(\vec{E}_\perp + \frac{\vec{v}}{c}\times\vec{B}_\perp\right)$$

$$E_3' = E_3 \qquad \vec{E}_\parallel' = \vec{E}_\parallel$$

$$B_1' = \gamma(B_1 + \beta E_2), \quad B_2' = \gamma(B_2 - \beta E_1) \quad \Big\} \quad \vec{B}_\perp' = \gamma\left(B_\perp - \frac{v}{c}\times E_\perp\right)$$

$$B_3' = B_3 \qquad \vec{B}_\parallel' = \vec{B}_\parallel$$

e) Instead of field tensor $\underline{\underline{F}}$, one can work with

invariants

$F_{\mu\nu}F_{\mu\nu} = \mathrm{Tr}\,\underline{\underline{F}}^2$

$G_{\mu\nu}G_{\mu\nu} = \mathrm{Tr}\,\underline{\underline{G}}^2$

$G_{\mu\nu}F_{\mu\nu} = \mathrm{Tr}\,\underline{\underline{F}}\,\underline{\underline{G}}$

$$\underline{\underline{G}} = \begin{pmatrix} 0 & -E_3 & E_2 & -iB_1 \\ E_3 & 0 & -E_1 & -iB_2 \\ -E_2 & E_1 & 0 & -iB_3 \\ iB_1 & iB_2 & iB_3 & 0 \end{pmatrix}$$

(Taken $\underline{E}$ & replace $\vec{E}\to\vec{B}$; $\vec{B}\to -\vec{E}$

G satisfies the eq: $\dfrac{\partial G_{\mu\nu}}{\partial X_\nu} = 0$)

Homogeneous pair of Maxwell Eqns

$$G_{\mu\nu}G_{\mu\nu} = -F_{\mu\nu}F_{\mu\nu} = 2(E^2 - B^2)$$

$$G_{\mu\nu}F_{\mu\nu} = 4\vec{E}\cdot\vec{B}$$

Both are invariants

Prove this to yourself

For EM waves both invariants are 0

Corollary: if $|E| > |B|$ in one reference frame $|E| > |B|$ in all reference frames.

(Same for $=, <$)

If there is a reference frame w/ $\vec{E} = 0, B \neq 0$ then there is no reference frame w/ $\vec{E} \neq 0$ $\vec{B} = 0$ & vice versa

Ex: a) Electric field of point charge moving at $\vec{V} = V\hat{e}_3$

In rest frame $\vec{E} = q\dfrac{\vec{r}}{r^3}$

In frame moving w/ velocity $-V\hat{e}_3$ w/respect to rest frame (so that charge has velocity $V\hat{e}_3$ in that frame)

$$E_1' = \gamma E_1 \qquad X_1 = X_1'$$

$$E_2' = \gamma E_2 \qquad X_2 = X_2'$$

$$E_3' = E_3 \qquad X_3 = \gamma\left(X_3' - \frac{vt'}{c^2}\right)$$

also need to transform coords.

we want fields at $t' = 0$ when origins of two frames coincide

$$r = \sqrt{x_j x_j}$$

$$= \sqrt{x_1'^2 + x_2'^2 + \gamma^2 x_3'^2}$$

$$\vec{r} = \begin{pmatrix} x_1' \\ x_2' \\ \gamma x_3' \end{pmatrix}$$

$$E' = \begin{pmatrix} \gamma E_1 \\ \gamma E_2 \\ E_3 \end{pmatrix} = \frac{\gamma q \vec{x}'}{\left(\sqrt{x_1'^2 + x_2'^2 + \gamma^2 x_3'^2}\right)^3}$$

in polar coords for K'

$$x_3' = r' \cos\theta$$

$$x_1'^2 + x_2'^2 + x_3'^2 = r'^2$$

$$x_1'^2 + x_2'^2 + \gamma^2 x_3'^2 = r'^2 + (\gamma^2 - 1) r'^2 \cos^2\theta$$

$$= \gamma^2 r'^2 (1 - \beta^2 \sin^2\theta)$$

$$\vec{E}' = \frac{q(1-\beta^2)\vec{r}'}{r'^3 (1 - \beta^2 \sin^2\theta)^{3/2}}$$

(Same result as derived in Chap 8)

b) The magnetic field of a charge moving at const velocity is:

$$B_1' = -\gamma\beta E_2$$ — Sign since K' moves at velocity $-V\hat{e}_3$

$$B_2' = \gamma\beta E_1$$

$$B_3' = 0$$

$\Rightarrow$ Magnetic field is in the $+\theta$ direction, with Magnitude

$$B_\theta' = \gamma\beta\sqrt{E_1^2 + E_2^2} = \frac{\gamma\beta\sqrt{x_1'^2 + x_2'^2}}{(x_1'^2 + x_2'^2 + \gamma^2 x_3'^2)^{3/2}}$$

Lets apply these results to a moving array of charges placed along the x_3' axis w/ charge density $\rho_e' dx_3' = \rho_e dx_3$ per unit length

Each element dx_3' gives a B-field along $\hat{e}_\theta$. Therefore

$$B_\theta' = \int dx_3' \rho_e' \frac{\gamma\beta\sqrt{x_1'^2 + x_2'^2}}{(x_1'^2 + x_2'^2 + \gamma^2 x_3'^2)^{3/2}}$$

$$= \frac{2\beta\rho_e'}{\sqrt{x_1'^2 + x_2'^2}} = \frac{2I'/c}{r_0'}$$

$r_0' = \sqrt{x_1'^2 + x_2'^2}$ = distance to x_3' axis

c) Radiated Power of accelerated charge

for $\beta \ll 1$: $P = \dfrac{2q^2a^2}{3c^2}$ (Larmor formula)

transform to reference frame at which β is no longer small. What is a in moving fr

recall $\underline{U} = (\gamma\vec{u}, \gamma i c)$

$$P' = -\frac{dW'}{dt'} \quad W' = \gamma W, \ldots$$

$$= -\frac{dW}{dt} = 2q^2a^2/3c\ldots$$

define $\underline{D} = \dfrac{d\underline{U}}{d\tau}$ how is $\underline{D}$ related to a?

use $d\tau = dt\sqrt{1 - u^2/c^2} \Rightarrow D = \dfrac{1}{\sqrt{1-\frac{u^2}{c^2}}} \dfrac{d}{dt}(\gamma u, i\ldots$

$$\underline{D} = \left(\frac{\dot{\vec{u}}}{1 - \frac{u^2}{c^2}} + \frac{\vec{u}(\vec{u}\cdot\dot{\vec{u}})}{c^2\left(1 - \frac{u^2}{c^2}\right)^2}, \; i\frac{\vec{u}\cdot\dot{\vec{u}}}{c\left(1 - \frac{u^2}{c^2}\right)^2} \right)$$

where $\dot{\vec{u}} = \dfrac{d\vec{u}}{dt} = \vec{a}$

In order to find $\vec{a}$ we write

$$\underline{D} = \left(\frac{\dot{\vec{u}}}{1 - \frac{u^2}{c^2}} + \frac{-i\vec{u}D_4}{c}, \; D_4 \right)$$

in rest frame: $\vec{u} = 0 \Rightarrow \underline{D} = (\vec{a}, 0)$

in moving frame K' (velocity $-v\hat{e}_3$ in K)

$D_1' = D_1 = u_1 = a_1$

$D_2' = D_2 = u_2 = a_2$

$D_3' = \gamma D_3 = \gamma u_3 = \gamma a_3$

$D_4' = i\beta\gamma D_3 = i\beta\gamma a_3$

2) In K' velocity $\vec{u}' = v\hat{e}_3$, hence

~~Substituting expressions ...~~

$D_1' = a_1'/(1-\beta^2)$

$D_2' = a_2'/(1-\beta^2)$

$D_3' = a_3'/(1-\beta^2) + \beta^2 a_3'/(1-\beta^2)^2$

$\Rightarrow D_3' = a_3'/(1-\beta^2)^2$

$D_4' = i\vec{\beta}\cdot\vec{a}'/(1-\beta^2)^2$

4) $a^2 = a_1^2 + a_2^2 + a_3^2 = \dfrac{a_1'^2 + a_2'^2}{(1-\beta^2)^2} + \dfrac{a_3'^2}{(1-\beta^2)^3}$

Power radiated:

$P' = -\dfrac{dW'}{dt'} = -\dfrac{dW}{dt}$

because $\left\{ \begin{array}{l} W' = \gamma W \\ dt' = \gamma dt \end{array} \right.$

$P' = \dfrac{2q^2 a^2}{3c^3} = \dfrac{2q^2}{3c^3}\dfrac{1}{(1-\beta^2)^2}\left(a_1'^2 + a_2'^2 + \dfrac{a_3'^2}{(1-\beta^2)}\right)$

if $\vec{V}$ and $\vec{a}$ are collinear

$a_{1,2} = 0$ reproduces known formula. if $\vec{V} \perp \vec{a}$ $a_3 = 0$

... reproduces known formula

$\underline{D}$ transforms as a four vector (1)

$D_1' = D_1$

$D_2' = D_2$

$D_3' = \gamma\left(D_3 + \frac{v}{c} D_4\right)$

$D_4' = \gamma\left(D_4 - \frac{v}{c^2} D_3\right)$

Since K is rest frame

$D_1 = a_1$

$D_2 = a_2$

$D_3 = a_3$

$D_4 = 0$

(3) applying transformation rules for $\underline{D}$:

$a_1 = a_1'/(1-\beta^2)$

$a_2 = a_2'/(1-\beta^2)$

$a_3 = a_3'/(1-\beta^2)^{3/2}$

Eq. of motion for a charged particle:

In rest frame K: $\dfrac{d\vec{p}}{dt} = qE$

$\dfrac{dW}{dt} = 0$

now look at frame K' moving at velocity $-v\hat{e}_3$ w.r.t K (so that particle moves at velocity $v\hat{e}_3$)

in K': $q' = q$ charge is invariant

$$p_1' = p_1,\quad p_2' = p_2,\quad p_3' = \gamma p_3 + \gamma\beta\frac{W}{c},\quad W' = \gamma W + \gamma\beta p_3 c \iff p_1 = p_1',\quad p_2 = p_2',\quad p_3 = \gamma p_3' - \gamma\beta\frac{W'}{c},\quad W = \gamma W' - \gamma\beta p_3' c$$

take time derivative to proper time

$d\tau = dt = dt'\sqrt{1-\beta^2}$

also use $\begin{cases} \vec{E}_\perp = \gamma(\vec{E}'_\perp + \frac{\vec{v}}{c}\times\vec{B}') \\ \vec{E}_\parallel = \vec{E}'_\parallel \end{cases}$

Then:

$$\frac{dp_1}{dt} = qE_1 = \frac{dp_1}{d\tau} = \gamma\frac{dp_1'}{dt'}$$

$$= q\gamma(\vec{E}' + \tfrac{1}{c}\vec{v}\times\vec{B}')_1$$

$$\frac{dp_2}{dt} = qE_2 = \frac{dp_2}{d\tau} = \gamma\frac{dp_2'}{dt'}$$

$$= q\gamma(\vec{E}' + \tfrac{1}{c}\vec{v}\times\vec{B}')_2$$

$$\frac{dp_3}{dt} = qE_3 = \frac{dp_3}{d\tau} = \frac{d}{d\tau}\left[\frac{p_3'}{\gamma} - \beta\frac{W}{c}\right]$$

$$= \frac{dp_3'}{dt'} = qE_3'$$

(We used: $\frac{dW}{d\tau} = 0$ & that time deriv. don't act on γ, β because they are a property of the frames K, K' not of the particle)

$$\frac{dW}{dt} = 0 = \frac{d}{d\tau}\left[\gamma W' - \gamma\beta p_3' c\right]$$

$$\Rightarrow \frac{dW'}{dt'} = v\frac{dp_3'}{dt'} = qvE_3'$$

Conclusion

$$\frac{d\vec{p}'}{dt'} = q\left(\vec{E}' + \frac{\vec{v}'}{c}\times\vec{B}'\right)$$

in all V'

One can rewrite the four equations:

$$\frac{d\vec{p}'}{d\tau} = \gamma(\vec{E}' + \vec{\beta}\times\vec{B}')$$

$$\frac{dW'}{d\tau} = \gamma c q\vec{\beta}\cdot\vec{E}'$$

or

$$\frac{dp'_\mu}{d\tau} = \underbrace{\frac{q}{c}F'_{\mu\nu}u'_\nu}_{\text{force in four vector notation}}$$

For a continuous charge density need "force density":

$$K'_\mu = \frac{1}{c}F'_{\mu\nu}\Big(\sum_{\substack{j \\ \text{particles}}} q\,u'_\nu(j)\Big) = \frac{1}{c}F'_{\mu\nu}J$$

using maxwell's eqs:

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{4\pi}{c}J_\mu \Rightarrow K_\mu = \frac{1}{4\pi}F_{\mu\nu}\frac{\partial F_{\nu\sigma}}{\partial x_\sigma}$$

$$= \left(\rho(\vec{E} + \tfrac{1}{c}\vec{u}\times B), \tfrac{i}{c}\vec{E}\cdot\vec{J}\right)$$

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(4) Energy Momentum Tensor

Generalization of energy density:

- $\mathcal{E} = \frac{1}{8\pi}(E^2 + B^2)$
- Poynting vector $\vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{B})$
- Maxwell Stress Tensor $\overleftrightarrow{T}$

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\sigma} F_{\sigma\nu} + \frac{1}{4}\delta_{\mu\nu} F_{\lambda\rho} F_{\lambda\rho}\right)$$

Explicitly:

$$T_{jk} = \frac{1}{4\pi}\left(E_j E_k + B_j B_k - \frac{1}{2}\delta_{jk}\left(E^2+B^2\right)\right) = \text{maxwell stress tensor}$$

$$T_{4k} = T_{k4} = \frac{-i}{4\pi}(\vec{E}\times\vec{B})_k = -\frac{i}{c}\vec{S}$$

$$T_{44} = \frac{1}{8\pi}\left(E^2+B^2\right) = \mathcal{E}$$

Energy & momentum Conservation

$$\frac{d\mathcal{E}}{dt} + \vec{\nabla}\cdot\vec{S} = -\vec{J}\cdot\vec{E}$$

$$\frac{1}{c^2}\frac{\partial\vec{S}}{\partial t} + \vec{\nabla}\cdot(-\vec{T}) = -K$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\text{momentum}\\ \text{density of}\\ \text{field}\end{pmatrix} + \nabla\cdot\begin{pmatrix}\text{momentum}\\ \text{current density}\\ \text{of field}\end{pmatrix} = -\text{ force density on charges/currents}$$

$$\frac{\partial T_{\mu\nu}}{\partial X_\nu} = K_\mu = \frac{1}{c} F_{\mu\nu} J_\nu$$

derivation:
$$\frac{\partial T_{\mu\nu}}{\partial X_\nu} = \frac{1}{4\pi}\left[\frac{\partial F_{\mu\sigma}}{\partial X_\nu} F_{\sigma\nu} + F_{\mu\sigma}\frac{\partial F_{\sigma\nu}}{\partial X_\nu} + \frac{1}{4}\delta_{\mu\nu}\frac{\partial}{\partial X_\nu}\left(F_{\lambda\rho} F_{\lambda\rho}\right)\right]$$

use:
$$\frac{\partial F_{\mu\sigma}}{\partial X_\nu} F_{\sigma\nu} = \frac{\partial F_{\sigma\mu}}{\partial X_\nu} F_{\nu\sigma} \quad \begin{pmatrix}F \text{ is}\\ \text{anti}\\ \text{symmetric}\end{pmatrix}$$

$$\underset{\nu\leftrightarrow\sigma}{=} \frac{\partial F_{\nu\mu}}{\partial X_\sigma} F_{\sigma\nu}$$

$$= \frac{1}{2}\left(\frac{\partial F_{\mu\sigma}}{\partial X_\nu} + \frac{\partial F_{\nu\mu}}{\partial X_\sigma}\right) F_{\sigma\nu}$$

use $\frac{\partial F_{\mu\sigma}}{\partial X_\nu} + \frac{\partial F_{\nu\mu}}{\partial X_\sigma} + \frac{\partial F_{\sigma\nu}}{\partial X_\mu} = 0$

$$\Rightarrow = -\frac{1}{2}\left(\frac{\partial F_{\sigma\nu}}{\partial x_\mu}\right)F_{\sigma\nu} = -\frac{1}{4}\frac{\partial}{\partial x_\mu}\left(F_{\sigma\nu}F_{\sigma\nu}\right)$$

$$\Rightarrow \frac{\partial T_{\mu\nu}}{\partial x_\nu} = \frac{1}{4\pi} F_{\mu\sigma}\frac{\partial F_{\sigma\nu}}{\partial x_\nu} = k_\mu$$