Lect 2)

1) Electrostatics

Formulation Possible in terms of
- Point charges q
- 1d Charge per length $\rho_e \, dl$
- 2d Charge per area $\rho_s \, da$
- 3d Charge per volume $\rho \, dV$

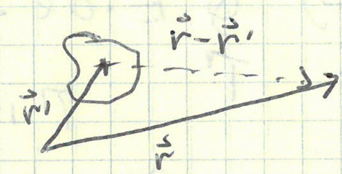
We'll take the 3d charge per volume approach

Main Problem: $\rho(\vec{r})$ given, what is $\vec{E}$?

Different approaches:

a) Coulomb field law

$$\vec{E}(\vec{r}) = \int_V dV' \frac{\rho(\vec{r}\,')\, \hat{e}_{\vec{r}-\vec{r}\,'}}{|\vec{r}-\vec{r}\,'|^2}$$



b) Gauss' law $\oint_S \vec{E}\cdot d\vec{a} = 4\pi \int_V dV \rho(\vec{r})$

tells us That we should be able to tell how much charge there is by looking at the flux of field lines through a surface.

Ex: take a Point charge at origin

$$\oint \vec{E}\cdot d\vec{a} = \int \frac{q}{r^2}\hat{r}\cdot\left(r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}\right) = 4\pi q$$ why no distance dependence? $\frac{1}{r^2}$ law

i) didn't have to be a sphere. ii) Can add any # of charges according to superposition.

$$\oint_S \vec{E}\cdot d\vec{a} = 4\pi Q_{enc.} = 4\pi \int_V dV \rho(\vec{r})$$

S: Closed Surface, $d\vec{a}$ outward pointing normal

V: enclosed volume

Most useful for:

1) Spherical Symmetry : make S a Concentric sphere
2) Cylindrical Symmetry : make S a Coaxial cylinder
3) plane Symmetry : use a pill box that straddles the surface

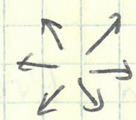
Using the Divergence theorem:

$$\oint_S E \cdot da = \int_V \vec{\nabla} \cdot \vec{E}\, dv = 4\pi \int \rho(\vec{r})\, dv$$

$$\text{div}\,\vec{E} = \vec{\nabla} \cdot \vec{E} = 4\pi\rho$$

Ex: draw field lines where $\vec{\nabla} \cdot \vec{E} = 0$, $\vec{\nabla} \cdot \vec{E} \neq 0$

$\vec{\nabla} \cdot \vec{E} = 0$



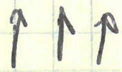
$\vec{\nabla} \cdot \vec{E} \neq 0$

c) $\oint_\Gamma E \cdot dl = 0$ Conservative field (not double valued) like gravity

Γ: is a closed Contour

Using Stokes' thm

$$\text{rot}\,\vec{E} = \text{curl}\,\vec{E} = \vec{\nabla} \times \vec{E} = 0$$



$\vec{\nabla} \times \vec{E} = 0$

$\vec{\nabla} \times \vec{E} \neq 0$

$\vec{\nabla} \times \vec{E} = 0$ at center & not at sides

$\vec{\nabla} \times \vec{E} = 0$

$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$, $\vec{\nabla} \times \vec{E} = 0$ are equivalent to Columbs law plus superposition

d) © implies there exists a scalar potential Φ

Such that $\vec{E} = -\text{grad}\,\Phi = -\nabla\Phi$

or $\Phi = -\int_{r_0}^{r} \vec{E} \cdot d\vec{l}$ reference point r_0 is arbitrary

Φ is only unique up to a Constant

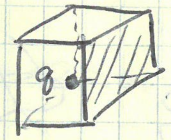
e) div grad $\Phi = \nabla^2 \Phi = -4\pi\rho$ Poisson's eq

$\nabla^2 \phi = 0$ charge free space Laplace's eq

Solution $\Phi(\vec{r}) = \int_V \frac{dv'\,\rho(r')}{|r-r'|} + \text{Const}$

any of these are good methods for backing out $\vec{E}$
Use the one that is most convenient. If symmetry is not spherical, cylindrical, or planar, find ϕ first then
$\vec{E} = -\vec{\nabla}\Phi$

Caution
Sometimes even a
Cube can have
spherical symmetry
for a specific problem



find flux through surface

2) Static Magnetic Fields

Formulation is possible in terms of:

- 1d Current element along wire $I\,d\vec{\ell}$
- 2d Current per width on a surface $\vec{K}\cdot da$
- 3d Current per area in a volume $\vec{J}\cdot dv$

well take the 3d current per area approach

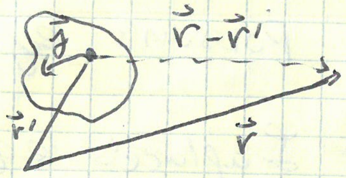
main problem: Static current density $\vec{j}\,dv$ given
What is $\vec{B}$?

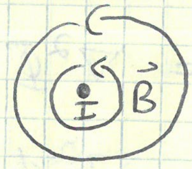
Static current density: all currents must flow in loops.

a) Biot-Savart law $\vec{B} = \frac{1}{c}\oint_V dv' \frac{j(r') \times \hat{e}_{\vec{r}-\vec{r}'}}{|\vec{r}-\vec{r}'|^2}$

integral is over volume in which current flows

Symbol $\oint$ indicates that current flows in loops





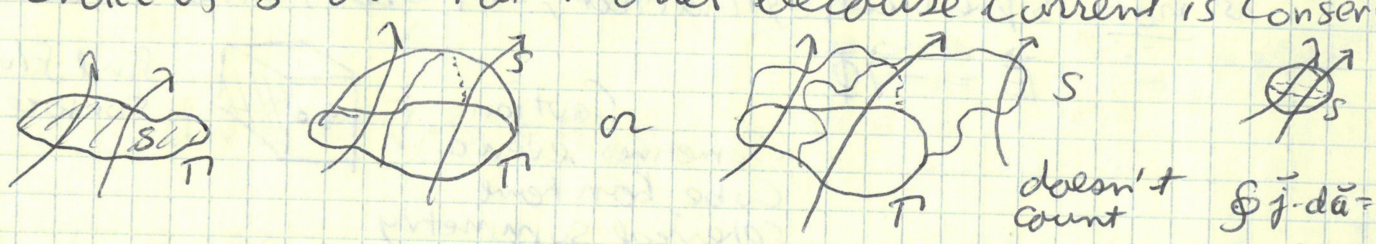
$$\oint \vec{B}\cdot d\vec{l} = \frac{4\pi}{c}\frac{I}{2\pi r}\oint d\vec{l} = \frac{4\pi}{c} I_{li}$$

b) $$\oint_{\Gamma} \vec{B}\cdot d\vec{l} = \frac{4\pi}{c} I_{link} = \frac{4\pi}{c}\oint_{S} \vec{j}\cdot d\vec{a}$$ ampere's law

Γ: closed contour

S: enclosed surface

Choice of S does not matter because current is conserved



any current in must also leave & go out

$\oint_S \vec{j}\cdot d\vec{a}=0$ is S is a closed surface.

Useful for special symmetries

Cylindrical, plane, solenoidal, toroidal

Using Stokes' thm

$$\text{Curl } \vec{B} = \vec{\nabla}\times\vec{B} = \frac{4\pi}{c} j$$

c) $\text{div } \vec{B} = \vec{\nabla}\cdot\vec{B} = 0$ "no magnetic monopoles"

$$\vec{\nabla}\cdot\vec{B} = \frac{1}{c}\int \nabla\cdot\frac{\vec{j}(r')\times\hat{e}_{r-r'}}{|r-r'|^2}dv = 0$$

$$\nabla\cdot\left(j\times\frac{\hat{e}_{rel}}{r^2}\right) = \frac{\hat{e}_{rel}}{r^2}\underbrace{\left(\nabla\times\vec{j}(r')\right)}_{0} - \vec{j}\underbrace{\left(\nabla\times\frac{e_r}{r^2}\right)}_{0}$$

since $\vec{j}$ is a function of r' not r

0 problem 1.62 in Griffiths

© implies there exists a vector field $\vec{A}$ such that $\vec{B} = \text{Curl}\,\vec{A} = \vec{\nabla} \times \vec{A}$

$\vec{A}$ is <u>not</u> uniquely determined by $\vec{B}$

a <u>possible</u> choice in terms of the currents that generate $\vec{B}$ is $\quad \vec{A} = \frac{1}{c} \oint_V \frac{\vec{j}(\vec{r}')\,dV'}{|\vec{r}-\vec{r}'|}$

$\vec{A}$ is not as useful as Φ. It's a vector, & still difficult to calculate. It will be very useful for electrodynamics

$\vec{\nabla} \cdot \vec{B} = 0$ & $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} j$ are equivalent to Biot-Savart law + superposition

3) Force law

$$\vec{F} = q\left(\vec{E} + \frac{\vec{u}}{c} \times \vec{B}\right) \qquad \text{Lorentz force}$$

force on a point particle with charge q & velocity $\vec{u}$

force on a current element: $dF = \frac{\vec{j}\,dv \times \vec{B}}{c}$

4) Relation between $\vec{E}$ & $\vec{j}$ in a Conductor

$$\vec{j} = \sigma \vec{E} \qquad \sigma: \text{Conductivity} \quad \text{Ohm's law}$$

<u>Problems:</u> ① Use Biot-Savart law to find the magnetic field $B_z(z)$ on the axis of a Circular loop carrying current I. Then show that $\int_{-\infty}^{\infty} B_z(z)\,dz = \frac{4\pi I}{c}$

② metal sphere in a uniform electric field draw field lines both inside & outside the sphere.