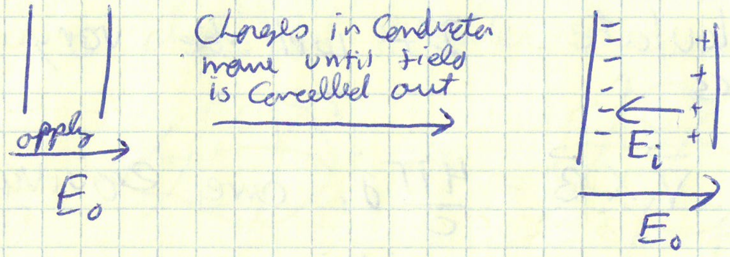
Lect 3 Media & $\vec{E}$ fields

0) $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ $\qquad \oint_S \vec{E}\cdot d\vec{a} = 4\pi\int \rho\, dv$

$\vec{\nabla}\times\vec{E} = 0$ $\qquad \oint_\Gamma \vec{E}\cdot d\vec{\ell} = 0$

1) <u>Conductors</u>

a) $E = 0$ inside a Conductor



Charge Continues to flow until Cancellation is Complete

process is practically instantaneously

b) $\rho = 0$ inside a Conductor

$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ is $\vec{E} = 0$ $\rho = 0$

c) Any net Charge is on the surface

d) a Conductor is an equipotential

$\phi = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{\ell}$ $\qquad$ Since $\vec{E} = 0$ inside a conductor

$\phi = 0$ $\qquad$ no potential change between $\vec{r}_0$ & $\vec{r}$

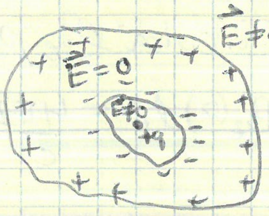
e) E is $\perp$ to surface just outside the Conductor

Otherwise charge will flow along surface until it Kills off the tangential component

f) induced Charges: ex1)

induced charge config. causes attraction

ex 2)



$\vec{E} \neq 0$ Conductor Communicates The presence of charge q to The outside world

ex 3) Cavity in a sphere ~~spherical~~

$\vec{E} = \frac{q}{r^2}\hat{r}$

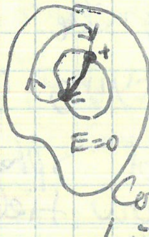
$\vec{r}$

Conductor Conceals info about shape of Cavity

ex 4) If a Cavity is empty of Charge field inside Cavity is Zero. No Charges inside so field lines would need to begin & end on Cavity wall.

$\int \vec{E}\cdot d\vec{\ell} = 0$

$E_{cavity} + \cancel{E_{metal}}^{0} = 0$



Safe inside a Car during Lightning

Make field line a part of loop going through part of conductor

1) Dielectric Media

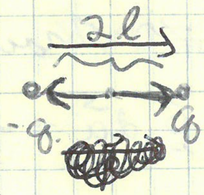
a) Charged particles are everywhere in nature
- Conduction electrons in metals
- molecules eg H_2O, etc.
- atoms 

Most charges are "bound." + & − charges can't be separated very far from one another by a typical force. Still, they CAN be separated by a short microscopic distance and or oriented (think dipoles in an $\vec{E}$ field)

Goal: Develop a macroscopic theory for the electric field that involves the free charges only (the ones we control)

Coarse Graining: average over distance/time scales that are:
$\gg$ microscopic distance/time scales
$\ll$ macroscopic distance/time scales where $\vec{E}$ varies

b) Definitions: electric dipole moment of a pair of bound charges $q, -q$ $\quad \vec{p} = 2q\vec{\ell}$ $\quad 2\vec{\ell}$: displacement vector from − charge to + charge.



c) Electric field may:
- induce dipole moments where they did not exist before
  ex: polarize an atom
- Align pre existing moments

to create an average or Net dipole moment per unit vol

"Polarization" $\vec{P} = \frac{1}{V} \sum_{\substack{\text{dipoles } j \\ \text{in } V}} \vec{p}_j$

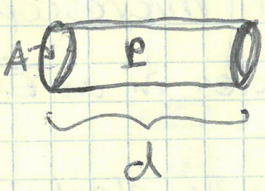
The same P can arise from a few large dipoles or many small ones different orientations. for our theory we don't care.

d) Luckily we don't have to keep track of all the dipoles. imagine a uniform string of dipoles

the net charge at the ends is called the "Bound" Charge since it can't be removed.

Ex: Tube with polarization $\underline{P}$ along axis

Parallel Cut

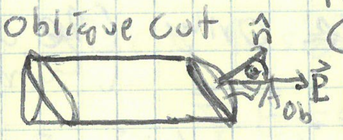


dipole moment $p = \underline{P}\, V_{volume} = \underline{P}(Ad) = q_{ends}\, d$

$q = \underline{P}A$ $\quad (\rho_s)_b = \dfrac{q}{A} = \underline{P}$

oblique cut

Ex:

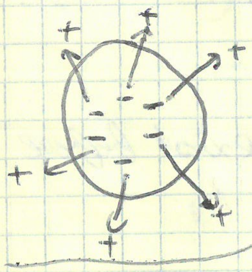


Polarization & Charge are the same but area is larger

$A = A_{ob}\cos\theta \qquad (\rho_s)_b = \dfrac{q}{A_{ob}} = \underline{P}\cos\theta = \vec{P}\cdot\hat{n}$

e) If Polarization is not uniform we may also accumulate bound charges within the material

net bound charge $\int \rho_b\, dv$ in a volume is equal & opposite to charge that has been "Pushed out" through surface $\vec{P}\cdot\hat{n}$ per unit area so $\int_V \rho_b\, dv = -\int_S \vec{P}\cdot d\vec{a} = -\int_V \vec{\nabla}\cdot\vec{P}\, dv$



$\rho_b = -\vec{\nabla}\cdot\vec{P}$

Alternative derivation:

$\Phi(r) = \int_V \dfrac{\hat{r}\cdot \underline{P}(r')}{r^2}\, dv' \qquad \vec{\nabla}'\left(\dfrac{1}{r}\right) = \dfrac{\hat{r}}{r^2}$

$\Phi(r) = \dfrac{\hat{r}\cdot\vec{p}}{r^2}$

$p = \int \underline{P}\, dv'$

$= \int_V \vec{P}\cdot\vec{\nabla}'\left(\dfrac{1}{r}\right) dv'$ integrate by parts

$= \int_V \nabla'\cdot\left(\dfrac{\underline{P}}{r}\right) dv' - \int_V \dfrac{1}{r}\left(\vec{\nabla}'\cdot\vec{P}\right) dv'$

$= \oint_S \underbrace{\dfrac{1}{r}\, \underline{P}\cdot d\vec{a}'}_{\text{Surface charge } (\rho_s)_b = \vec{P}\cdot da} - \underbrace{\int_V \dfrac{1}{r}\left(\vec{\nabla}'\cdot\vec{P}\right) dv'}_{\text{Potential due to volume charge } \rho_b = \vec{\nabla}'\cdot\vec{P}} = \oint_S \dfrac{(\rho_s)_b\, da'}{r} + \int_V \dfrac{\rho_b\, dv'}{r}$

e) with the free charges & the bound charges

$$\rho = \rho_f + \rho_b$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho = 4\pi\rho_f + 4\pi\rho_b$$

$$= 4\pi\rho_f - 4\pi\,\vec{\nabla}\cdot\vec{P}$$

define $\vec{D} = \vec{E} + 4\pi\vec{P}$ Dielectric displacement

Then, Gauss' law can be rewritten in a form that only contains the free charge density ρ_f

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho_f \quad \text{or} \quad \oint \vec{D}\cdot d\vec{a} = 4\pi\int_V \rho_f\, dV$$

It is tempting to think of $\vec{D}$ as just $\vec{E}$ for free charge ρ_f

Not true   no Coulomb's law for $\vec{D}$

$$\vec{D}(\vec{r}) \neq \int_V dV' \frac{\rho_f(\vec{r}')\,\hat{e}_{\vec{r}-\vec{r}'}}{|\vec{r}-\vec{r}'|^2}$$

D is not determined exclusively by the free charge

need both $\vec{\nabla}\cdot\vec{D}$ & $\vec{\nabla}\times\vec{D}$ to determine $\vec{D}$ field.

$$\vec{\nabla}\times\vec{D} = \underbrace{\cancel{\vec{\nabla}\times\vec{E}}}_{0} + 4\pi\,\vec{\nabla}\times\vec{P}$$

does not always vanish

$\vec{D} \neq -\nabla\Phi_f$ since $\vec{\nabla}\times\vec{D}$ is not always zero.

Strategy: for spherical, cylindrical, or plane symmetry

Can get $\vec{D}$ from $\oint \vec{D}\cdot d\vec{a} = 4\pi\int_V \rho_f\, dV$

Otherwise you need to use a different approach.

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note: in metals $\vec{P} = 0$ $\vec{\nabla}\times\vec{P} = 0$ so $\vec{\nabla}\times\vec{D} = 0$ & $\vec{\nabla}\cdot\vec{D} = 0$

so $\vec{D} = 0$

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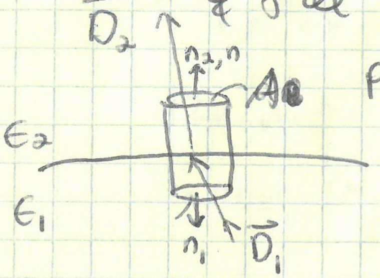
f) Linear Dielectrics for weak electric fields

$\vec{P} = \underbrace{\chi_e}_{\text{electric susceptibility}} E$ $\vec{D} = \epsilon\vec{E}$ $\epsilon = 1 + 4\pi\chi_e$ dielectric constant

$\epsilon = 1$ in vacuum

g) Boundary Conditions between two different media 1, 2 w/ different dielectric constants & free surface charge (need not be linear dielectrics)



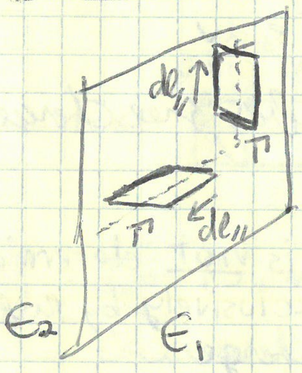
Planar geometry

$$\int_S d\vec{a}\cdot\vec{D} = 4\pi\int_V dv\,\rho_f$$

Shrink pill box to zero height

$$\cancel{A}(\vec{D}_2-\vec{D}_1)\hat{n} = 4\pi(\rho_s)_f\cancel{A}$$

$$\boxed{(\vec{D}_2-\vec{D}_1)\cdot\hat{n} = 4\pi(\rho_s)_f}$$



Shrink part of loop that straddles surface

$$\oint_\Gamma \vec{E}\cdot d\vec{l} = 0 \quad = \vec{E}_1\cdot d\vec{l}_\parallel - \vec{E}_2\cdot d\vec{l}_\parallel = 0$$

Tangential Component of E is Continuous

Since $\Phi \equiv -\int_{r_0}^{r}\vec{E}\cdot d\vec{l}$ if we cross a surface $\int dl \to 0$

So $\Phi_1 = \Phi_2$ & Φ is Continuous

Magnetic Materials & $\vec{B}$ fields

0) $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}j$ $\quad\oint_\Gamma \vec{B}\cdot d\vec{l} = \frac{4\pi}{c}\int \vec{j}\cdot d\vec{a}$

$\vec{\nabla}\cdot\vec{B} = 0$ $\quad\int_S \vec{B}\cdot d\vec{a} = 0$

1) Magnetic Materials:

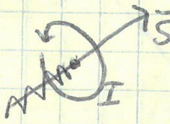
a) Nature is full of electrical currents
- electrons precessing around Nuclei
- Currents of Conduction e^- in metals

There are also magnetic moments not connected to currents (e.g. Spin)

Most currents are "bound" currents of microscopic size still, they can produce a net current density that needs to be accounted for our theory of materials

Goal: Develop a macroscopic theory for the magnetic field that involves free currents only

b) Definition: Magnetic moment of a current loop

$\vec{m} = \frac{I}{c}\vec{S}$ 

$\leftarrow$ area of loop (use Right hand rule for orientation)

Magnetization $\vec{M} = \frac{1}{V}\sum_{j \in V} \vec{m}_j$ magnetic moment per unit volume

c) For most materials $\vec{M} = 0$ if there is no applied field (exeption: ferromagnets, low temps, etc)

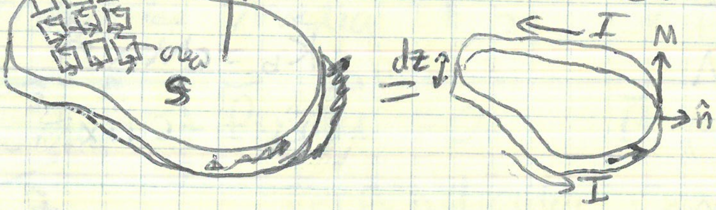
$\vec{B}$ may cause $\vec{M}$ to be non zero

- $\vec{B}$ induces magnetic moments – "diamagnetism" typically induced magnetic moments are anti parallel to B
- $\vec{B}$ aligns pre-existing magnetic moments "paramagnetism" Typically aligned moments are parallel to $\vec{B}$.

$\vec{M}$ can arise in many different ways (few large moments, many small ones, different orientations)
For our macroscopic theory the origin does not matter.

d) Luckily we don't have to keep track of all the magnetic moments. Imagine a thin slab of uniformly magnetized material.

all internal currents cancel accept at edge



$m = MS\,dz$
$\downarrow$ but also
$m = \frac{IS}{c}$ $\therefore$ $I = cM\,dz$
$\downarrow$
$K_b = cM$

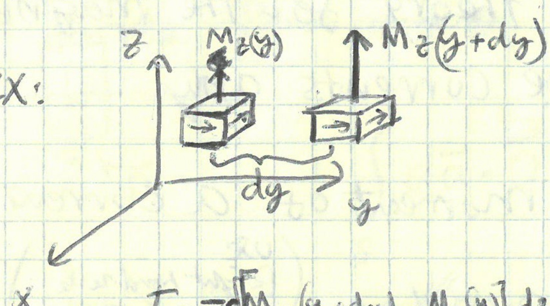
$(\vec{K}_b) = c\vec{M}\times\hat{n}$

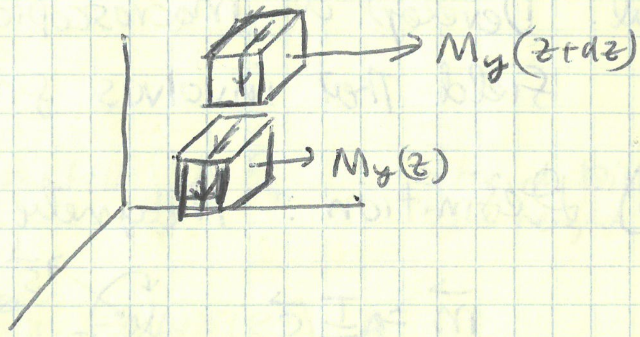
$$\boxed{\frac{K_b}{c} = -\hat{n}\times\vec{M}}$$ where $\hat{n}$ is outward normal

K_b is current per width of boundary

Nonuniform Magnetization leads to non-cancelling internal currents

ex:





$$I_x = c[M_z(y+dy) - M_z(y)]\,dz$$
$$= \frac{\partial M_z}{\partial y}\,dy\,dz$$

$$(J_b)_x = c\frac{\partial M_z}{\partial y} \qquad (J_b)_x = -c\frac{\partial M_y}{\partial z}$$

$$(J_b)_x = c\left[\frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}\right]$$ Similar arguments for y, z

$$\boxed{J_b = c\vec{\nabla}\times\vec{M}}$$

Alternative derivation

$$\vec{A}(\vec{r}) = \frac{1}{c}\,\frac{\vec{m}\times\hat{r}}{r^2}$$

$$\vec{A}(\vec{r}) = \frac{1}{c}\int \frac{\vec{M}(\vec{r}')\times\hat{r}}{r^2}\,dv' \qquad \nabla'\left(\frac{1}{r}\right) = \frac{\hat{r}}{r^2}$$

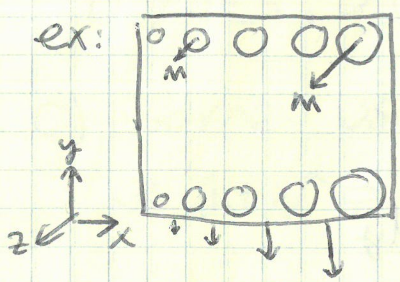
$$\vec{A}(\vec{r}) = \frac{1}{c}\int\left[\vec{M}(\vec{r}')\times\left(\nabla'\frac{1}{r}\right)\right]dv'$$ integrate by parts

$$= \frac{1}{c}\int\frac{1}{r}\left[\nabla'\times\vec{M}(\vec{r}')\right]dv' - \frac{1}{c}\int\nabla'\times\left(\frac{\vec{M}(\vec{r}')}{r}\right)dv'$$

$$= \frac{1}{c}\int \frac{1}{r}\underbrace{\left[\nabla'\times\vec{M}(\vec{r}')\right]}_{\text{volume current}}dv' + \frac{1}{c}\oint\frac{1}{r}\underbrace{\left[\vec{M}(\vec{r}')\times d\vec{a}'\right]}_{\text{surface current}}$$

$$\vec{J}_b = c\vec{\nabla}\times\vec{M} \qquad \vec{K}_b = c\vec{M}\times\hat{n}$$

$$\vec{K}_b = -c\,\hat{n}\times\vec{M}$$

ex: 

$\vec{M} = M_0 x^2 \hat{z}$ $\quad \vec{\nabla} \times \vec{M} = -2 M_0 x \hat{y}$

$\vec{j}_b = -2 c M_0 x \hat{y}$

d) with free currents & bound currents:

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}(j_b + j_f) = \frac{4\pi}{c} j_f + 4\pi \vec{\nabla} \times \vec{M}$$

define $\vec{H} = \vec{B} - 4\pi \vec{M}$ "H-field"

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c} j_f \qquad \oint \vec{H} \cdot d\vec{\ell} = I_{\text{free enclosed.}}$$

tempting to think of H as B for free ~~cu~~ currents.
<u>not the case</u> $\quad \vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}$

Strategy: for cylindrical, plane, solenoidal or toroidal symmetry use $\oint \vec{H} \cdot d\vec{\ell} = I_{f\,enc}$ otherwise we need a different approach.

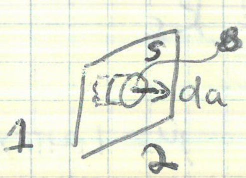
e) ~~Boundary Conditions between~~ Linear Magnetic Materials.

$\vec{M} = \chi_m H \qquad \chi_m \equiv$ "magnetic susceptibility"

$\vec{B} = \mu H \qquad \mu = 1 + 4\pi \chi_m$ magnetic permeability

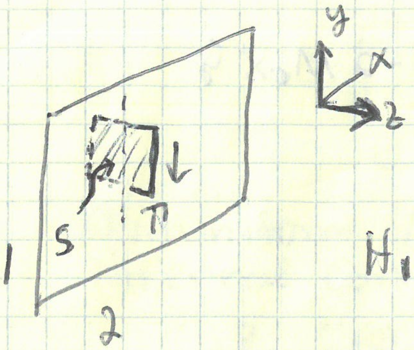
in vacuum $\mu = 1$

f) Boundary Conditions between two different magnetic media 1, 2 w/ free surface current density $\vec{K}_f$



$$\oint_S \vec{B} \cdot d\vec{a} = 0 \qquad \vec{B}_1 \cdot (-d\vec{a}) + \vec{B}_2 \cdot d\vec{a} = (\vec{B}_2 - \vec{B}_1) \cdot d\vec{a} = 0$$

Perpendicular Component of $\vec{B}$ is Continuous



$$\oint_{\Gamma} \vec{H}\cdot d\vec{l} = \frac{4\pi}{c}\int_S \vec{j}_f\, d\vec{a}$$

$$H_{1y}\,dy - H_{2y}\,dy = \frac{4\pi}{c}(j_f)_x\, dy$$

$$H_{1y} - H_{2y} = \frac{4\pi}{c}(j_f)_x$$

Similarly $H_{1x} - H_{2x} = -\frac{4\pi}{c}(j_f)_y$

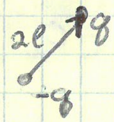
$$(\vec{H}_2 - \vec{H}_1)\times \hat{n}_{1\to 2} = -\frac{4\pi}{c}\vec{K}_f$$

Summary

E-fields

Bound Charges vs. free Charges

dipole moment $\vec{p} = 2\vec{l}q$



Polarization $\vec{P} = \frac{1}{V}\sum_{\substack{\text{dipoles}\\ j \text{ in } V}} \vec{p}_j$

Bound Charge density $\rho_b = -\vec{\nabla}\cdot\vec{P}$

Bound Surface charge density $(\rho_s)_b = \hat{n}\cdot\vec{P}$

Reformulation of Gauss' law

~~Boundary~~ $\vec{\nabla}\cdot\underbrace{(\vec{E}+4\pi\vec{P})}_{\vec{D}} = 4\pi\rho_f$

$\vec{D}$ dielectric displacement

Boundary between two dielectrics

$$(\vec{D}_2 - \vec{D}_1)\cdot\hat{n} = 4\pi(\rho_s)_f$$

$$(\vec{E}_2 - \vec{E}_1)\times\hat{n} = 0$$

Linear dielectric media

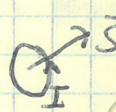
$\vec{P} = \chi_e\vec{E}$ $\quad\vec{D} = \epsilon\vec{E}$

$\epsilon = 1 + 4\pi\chi_e$

$\vec{B}$-fields

Bound Currents vs free Currents

Magnetic moment $\vec{m} = \frac{I}{c}\vec{S}$



Magnetization $\vec{M} = \frac{1}{V}\sum_j \vec{m}_j$

Bound Current density $\vec{j}_b = c\vec{\nabla}\times\vec{M}$

Bound Surface current $\vec{K}_b = -c\hat{n}\times\vec{M}$

Reformulation of Ampere's law

$$\vec{\nabla}\times\underbrace{(\vec{B}-4\pi\vec{M})}_{\vec{H}} = \frac{4\pi}{c}\vec{j}_f$$

$\vec{H}$: ~~[illegible]~~ "H field"

Boundary between two magnetic media

$$(\vec{B}_2 - \vec{B}_1)\cdot\hat{n} = 0$$

$$(\vec{H}_2 - \vec{H}_1)\times\hat{n} = -\frac{4\pi}{c}\vec{K}_f$$

Linear Mag media

$\vec{M} = \chi_m\vec{H}$ $\quad\vec{B} = \mu\vec{H}$ $\quad\mu = 1 + 4\pi\chi_m$