Dipoles, quadrupoles, Multipole expansions

0) Some useful Math relations

a) $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \dots$

b) $\vec{\nabla} r = \frac{\partial r}{\partial x}\hat{e}_x + \frac{\partial r}{\partial y}\hat{e}_y + \frac{\partial r}{\partial z}\hat{e}_z$ here r is the magnitude

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r} \text{ etc.}$$

$$\vec{\nabla} r = \frac{x}{r}\hat{e}_x + \frac{y}{r}\hat{e}_y + \frac{z}{r}\hat{e}_z = \frac{\vec{r}}{r} = \hat{e}_{\vec{r}}$$

c) $\vec{\nabla}(\vec{a}\cdot\vec{r}) = \vec{\nabla}(a_x x + a_y y + a_z z)$

$$= a_x\hat{e}_x + a_y\hat{e}_y + a_z\hat{e}_z$$

$$= \vec{a}$$

d) Taylor series of a function $f(x_1', x_2', x_3')$

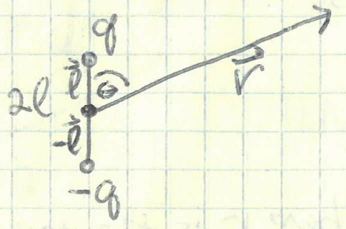
$$f(\vec{r}\,') = f(x_1', x_2', x_3') = f(0,0,0) + \sum_{i=1}^{3} x_i' \frac{\partial f}{\partial x_i'}\Big|_{\vec{r}\,'=0} + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} x_i' x_j' \frac{\partial^2 f}{\partial x_i' \partial x_j'}\Big|_{\vec{r}\,'=0}$$

Lets apply this to $f(x_1', x_2', x_3') = f(\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|}$

Use $\frac{\partial f}{\partial x_i'} = -\frac{\partial f}{\partial x_i}$, $\frac{\partial^2 f}{\partial x_i' \partial x_j'} = \frac{\partial^2 f}{\partial x_i \partial x_j}$

$$\frac{1}{|r-r'|} = \underbrace{\frac{1}{|\vec{r}|}}_{\text{related to monopoles}} \underbrace{- \sum_{i=1}^{3} x_i' \frac{\partial}{\partial x_i}\frac{1}{|\vec{r}|}}_{\text{related to dipoles}} + \underbrace{\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} x_i' x_j' \frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{|\vec{r}|}}_{\text{related to Quadrupoles}} + \dots$$

1) Electric field of a dipole



find field for $|\vec{r}| \gg |\vec{\ell}|$

$$\Phi(\vec{r}) = \frac{q}{|\vec{r}-\vec{\ell}|} - \frac{q}{|\vec{r}+\vec{\ell}|}$$

use $|\vec{r} \pm \vec{\ell}| = \sqrt{(r\cos\theta \pm \ell)^2 + (r\sin\theta)^2}$

$$= \sqrt{r^2 \pm 2r\ell\cos\theta + \ell^2}$$

$$= r\sqrt{1 \pm \frac{2\ell}{r}\cos\theta + \frac{\ell^2}{r^2}}$$

$$\frac{1}{|\vec{r} \pm \vec{\ell}|} = \frac{1}{r}\left(1 \pm \frac{2\ell}{r}\cos\theta + \frac{\ell^2}{r^2}\right)^{-1/2} \approx \frac{1}{r} \mp \frac{\ell}{r^2}\cos\theta$$

for $r \gg \ell$

$$\Phi(\vec{r}) = \frac{2q\ell}{r^2}\cos\theta + \mathcal{O}\left(\frac{q\ell^2}{r^3}\right)$$

in terms of dipole moment $\vec{p} = 2q\vec{\ell}$:

$$\Phi(\vec{r}) = \frac{\vec{p}\cdot\hat{e}_{\vec{r}}}{r^2}$$

$\hat{e}_{\vec{r}}$: unit vector in direction of $\vec{r}$

$\hat{e}_{\vec{r}} = \frac{\vec{r}}{r}$

$$\vec{E} = -\vec{\nabla}\Phi = -\vec{\nabla}\left(\frac{\vec{p}\cdot\vec{r}}{r^3}\right) = -\left(\vec{\nabla}\frac{1}{r^3}\right)(\vec{p}\cdot\vec{r}) - \frac{1}{r^3}\vec{\nabla}(\vec{p}\cdot\vec{r})$$

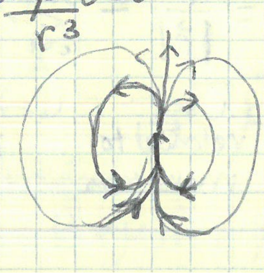
$$= \frac{3\vec{r}}{r^5}(\vec{p}\cdot\vec{r}) - \frac{1}{r^3}\vec{p} = \frac{1}{r^5}\left(3(\vec{p}\cdot\vec{r})\vec{r} - \vec{p}r^2\right)$$

For polar coords w/ $\vec{p}\cdot\hat{e}_{\vec{r}} = 2q\ell\cos\theta$

$$E_r = -\frac{\partial\Phi}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{2q\ell\cos\theta}{r^2}\right) = \frac{2\vec{p}\cos\theta}{r^3}$$

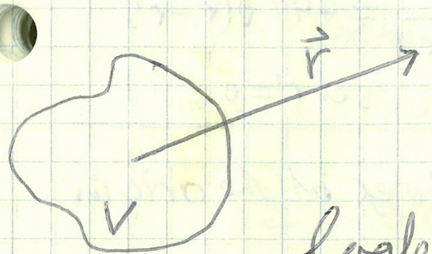
$$E_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = \frac{p\sin\theta}{r^3}$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi} = 0$$



2) Multipole Expansion

Arbitrary charge distribution of finite volume V



$$\Phi(\vec{r}) = \int_V dV' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

look at $\vec{r}$ far from V & choose origin inside V

then $|\vec{r}| \gg |\vec{r}\,'|$ This allows us to expand Φ in the small ratio $\frac{|\vec{r}\,'|}{|\vec{r}|}$

$$\Phi(\vec{r}) = \Phi^{(1)} + \Phi^{(2)} + \Phi^{(4)} + \dots$$

$$\Phi^{(1)}(\vec{r}) = \underbrace{\int_V dV' \rho(\vec{r}\,')}_{q:\ \text{total charge in V}} \frac{1}{|\vec{r}|}$$ monopole potential

"−" comes from $\frac{\partial}{\partial x_i} = -\frac{\partial}{\partial x_i'}$

$$\Phi^{2}(\vec{r}) = -\sum_{i=1}^{3} \int_V dV'\, x_i'\, \rho(\vec{r}\,') \frac{\partial}{\partial x_i} \frac{1}{|\vec{r}|}$$ dipole potential

$$\Phi^{(4)}(\vec{r}) = \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \int_V dV'\, x_i' x_j' \rho(\vec{r}\,') \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{|\vec{r}|}$$ quadrupole potential

$$\int_V dV' \rho(r') = q$$

$$\Phi^{(1)}(\vec{r}) = \frac{q}{r}$$ monopole potential

Define dipole moment $\vec{p}$ of charge distribution

$$\vec{p} = \int_V dV'\, \vec{r}\,' \rho(\vec{r}\,')$$ for point charges $\vec{p} = q\vec{\ell} - q(-\vec{\ell}) = 2q\vec{\ell}$

$$\Phi^{(2)} = -\sum_{i=1}^{3} p_i \frac{\partial}{\partial x_i} \frac{1}{r} = -(\vec{p}\cdot\vec{\nabla})\frac{1}{r} = \frac{\vec{p}\cdot\vec{r}}{r^3}$$

Leading contribution to the potential far away from a distribution w/ no net charge

Quadrupole potential

a) auxiliary Math result

$$\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} \frac{1}{r} = 0 \quad \text{if} \quad \mathbf{r} \neq 0$$

Greens function

$$\nabla^2 \frac{1}{|r-r'|} = -4\pi \delta(r-r')$$

set $r' = 0$

because $\frac{1}{r}$ is the potential of a point charge at the origin & hence satisfies Laplace's eq.

$$\Delta \frac{1}{r} \text{ or } \nabla^2 \frac{1}{r} = 0 \quad \text{away from the origin at } r=0$$

Can subtract any constant times $\nabla^2 \frac{1}{r}$ from $\Phi^{(4)}$, it not alter the result.

$$\sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r} = 0 \quad \text{if } r \neq 0$$

b) $$\Phi^{(4)}(\vec{r}) = \frac{1}{2} \sum_{i,j=1}^{3} \left[\int_V dv' \rho(\vec{r}') x_i' x_j' \right] \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r}$$

Subtract for $r \neq 0$ $$0 = \frac{1}{6} \sum_{i,j=1}^{3} \left[\int_V dv' \rho(\vec{r}') r'^2 \delta_{ij} \right] \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r}$$

$$\Phi^{(4)} = \frac{1}{6} \sum_{i,j=1}^{3} \left[\int_V dv' \rho(\vec{r}') \left(3x_i' x_j' - r'^2 \delta_{ij}\right) \right] \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r}$$

$$\equiv \frac{1}{6} \sum_{i,j=1}^{3} \underbrace{Q_{ij}}_{\text{deals with Charge distribution}} \underbrace{\frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r}}_{\text{deals with test point location}}$$

c) $$\frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r} = \frac{3x_i x_j - \vec{r}^2 \delta_{ij}}{r^5}$$

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}\left(\frac{1}{r}\right)\right) = \frac{\partial}{\partial x_i} \frac{-x_j}{\left(\sqrt{x_1^2+x_2^2+x_3^2}\right)^{3/2}}$$

if $x_i = x_j$: $-\frac{1}{r^3} + \frac{3x_j x_j}{r^5}$

if $x_i \neq x_j$: $\frac{3x_i x_j}{r^5}$

$$\frac{3x_i x_j}{r^5} - \frac{\delta_{ij}}{r^3}$$

d) • Q is a 3×3 matrix or tensor 9 elements

• Q is real & symmetric $Q_{ij} = Q_{ji}$ 6 elements

look at formula $x_i \to x_j$ keeps Q the same

• Q is traceless $\operatorname{tr} Q = \sum_{k=1}^{3} Q_{kk} = 0$ 5 elements

$$\sum_{i=j=1}^{3} 3x_i' x_j' - r'^2 \delta_{ij} = 0$$ (This is why we added the $\nabla^2 \frac{1}{r}$ term – to force trace to vanish)

Choose a basis in which Q is diagonal "principle axes"
This gets rid of off-diagonal terms.

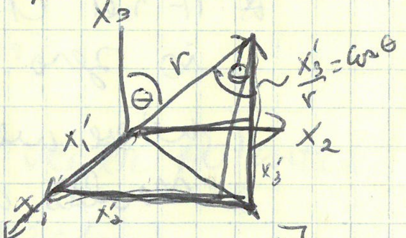
$$Q = \begin{pmatrix} Q_{11} & 0 & 0 \\ 0 & Q_{22} & 0 \\ 0 & 0 & Q_{33} \end{pmatrix} \qquad Q_{11} + Q_{22} + Q_{33} = 0$$

$\therefore$ Only two elements are independent

e) If there is enough symmetry sometimes you get lucky and two of the elements of the diagonalized Q are equal. So if $Q_{11} = Q_{22}$ Then one calls $Q_{33} \equiv Q$ The quadrupole moment.

$$Q_{33} = \int dv' \rho(r') \left(3x_3'^2 - r'^2\right) \text{ (primes)}$$ for this special case

not the same (no primes)

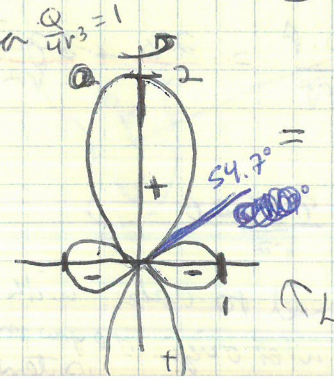


$$\Phi^{(4)}(\vec{r}) = \frac{1}{6r^5} \sum_{j=1}^{3} Q_{jj} \left(3x_j^2 - r^2\right)$$

Polar plot of Φ^4

$$= \frac{Q_{33}}{6r^5}\left[-\frac{1}{2}\left(3x_1^2 - r^2\right) - \frac{1}{2}\left(3x_2^2 - r^2\right) + \left(3x_3^2 - r^2\right)\right]$$

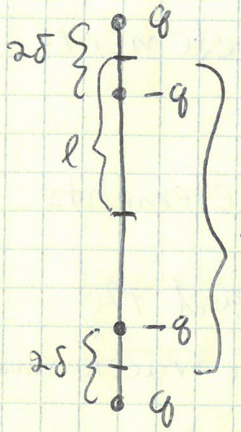
$$= \frac{Q_{33}}{4r^3}\left[-\frac{x_1^2}{r^2} + \frac{1}{3} - \frac{x_2^2}{r^2} + \frac{1}{3} + \frac{2x_3^2}{r^2} - \frac{2}{3}\right]$$



$$= \frac{Q_{33}}{4r^3}\left[\underbrace{-\frac{\left(x_1^2 + x_2^2 + x_3^2\right)}{r^2}}_{-1} + \frac{3x_3^2}{r^2}\right] = \frac{Q}{4r^3}\left(3\cos^2\theta - 1\right)$$

$\frac{Q_{33}}{4r^3}$ is fixed & here radius corresponds to amplitude of $\Phi^{(4)}$ &

Ex: Two Coaxial but oppositely oriented dipoles



net Charge $= 0$

net dipole moment $= 0$

rotational symmetry around 3rd axis

$Q_{11} = Q_{22}$ (lucky case)

$$Q_{33} = \sum_\alpha q_\alpha \, 3X_{3\alpha}^2 - r_\alpha^2$$

$$= 2\sum_\alpha q_\alpha X_{3\alpha}^2 \quad \text{since } r_\alpha = X_{3\alpha}$$

$$= 2q\left[(l+\delta)^2 - (l-\delta)^2 - (-l+\delta)^2 + (-l-\delta)^2\right]$$

$$= 16 q l \delta \qquad p = 2q\delta$$

$$= 8lp \equiv Q \qquad Q_{11} = Q_{22} = -\frac{Q}{2} = 4lp$$

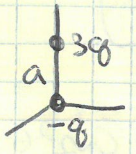
$$\Phi^{(4)}(\vec{r}) = \frac{Q}{4r^3}(3\cos^2\theta - 1) = \frac{2pl}{r^3}(3\cos^2\theta - 1)$$

* Notice that this picture is only valid for $\vec{r} \gg \vec{r}'$ if we look too close to the origin this picture breaks down.

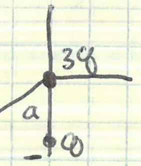
* If net charge is zero & (for Q) if the net dipole moment is zero too these results end up being indep of where we choose the origin. Math becomes trickier but results are the same.

* If net charge is not zero etc. then results are <u>not</u> the same

ex: Calculate monopole, dipole moments & approx potentials for



monopole: $Q = 2q$ dipole: $p = 3qa\hat{z}$ potential: $\Phi = \frac{Q}{r} + \frac{p\cdot\hat{r}}{r^2} = \frac{2q}{r} + \frac{3qa\cos\theta}{r^2}$

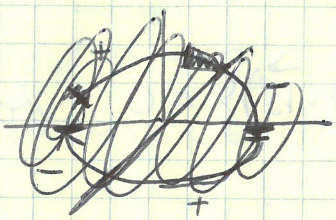


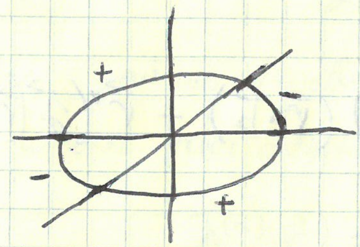
$Q = 2q \qquad p = qa\hat{z} \qquad \Phi = \frac{2q}{r} + \frac{qa\cos\theta}{r^2}$

unless all prior moments cancell you need to ask with respect to what origin are you measuring

One more example for quadrupole (for HW)

Consider a ring of radius a lying in the x-y ~~axis~~ plane with line charge $\rho_e = \pm\lambda$ for alternating segments

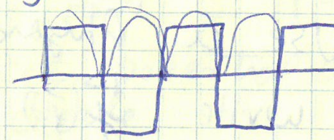




Some items suggest how to calc.

$$Q_{ij} = \int_0^{2\pi} \rho_e(\theta)\, a\, d\theta \left(3x_i' x_j' - r'^2 \delta_{ij}\right)$$

ρ is a step function



phase matters.

The analysis we just conducted in terms of moments ~~also~~ appears in many situations

Ex: Classical Mech. of Rigid bodies

mass $= \int_V dv'\, \rho_m(r')$ relevant for accel

moment of inertia

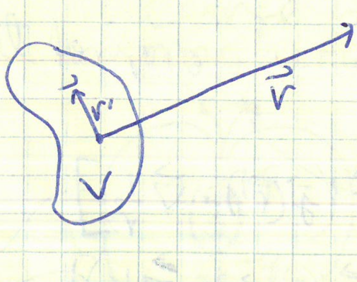
$$I_{ij} = \int_V dv' \left(r'^2 \delta_{ij} - x_i' x_j'\right) \rho_m(\vec{r}\,')$$

relevant for rotational acceleration

Lets repeat this analysis for the vector potential

1) Multipole expansion for vector potential

a) Current density $\vec{j}$ confined to finite volume V



$$\vec{A}(\vec{r}) = \frac{1}{c} \oint_V dv' \frac{\vec{j}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$$

use $$\frac{1}{|\vec{r} - \vec{r}\,'|} = \frac{1}{r} - \sum_{j=1}^{3} x_j' \frac{\partial}{\partial x_j} \frac{1}{r} + \cdots$$

$$= \frac{1}{r} - \vec{r}\,' \cdot \vec{\nabla} \frac{1}{r} + \cdots$$