b) A couple of Math results.

\[ \oint_S \mathbf{v} \cdot \mathbf{A}(\mathbf{r}') = 0 \]

\[ \oint_S \mathbf{v} \cdot \left\{ \mathbf{J}(\mathbf{r}) \left( \mathbf{r} \cdot \mathbf{A} \right) + \mathbf{v} \left( \mathbf{A}(\mathbf{r}') \cdot \mathbf{v} \right) \right\} = 0 \quad \text{for any } \mathbf{v} \]

Use a "physicist's proof."

Write \[ \mathbf{J}(\mathbf{r}) = \alpha_x \frac{d}{dt} \mathbf{r}_x \]

\[ \oint_S \mathbf{v} \cdot \left\{ \mathbf{J}(\mathbf{r}) \left( \mathbf{r}_x \cdot \mathbf{v} \right) + \mathbf{v} \left( \mathbf{A}(\mathbf{r}') \cdot \mathbf{v} \right) \right\} = 0 \quad \text{for stationary currents} \]

Currents have no end or beginning so an integral over a current loop gives zero just as much current flowing in one direction as in the other.

\[ \oint_S \mathbf{v} \cdot \left\{ \mathbf{J}(\mathbf{r}) \left( \mathbf{r}_x \cdot \mathbf{v} \right) + \mathbf{v} \left( \mathbf{A}(\mathbf{r}') \cdot \mathbf{v} \right) \right\} = 0 \quad \text{for stationary currents} \]

Same reasoning, integral over a current loop gives zero.

\[ \mathbf{A}(\mathbf{r}) = \mathbf{A}_i(\mathbf{r}) + \mathbf{A}_m(\mathbf{r}) + \ldots \]

\[ \mathbf{A}_i(\mathbf{r}) = \frac{1}{c} \left[ \oint_S \mathbf{v} \cdot \mathbf{J}(\mathbf{r}') \right] \frac{1}{r} = 0 \quad \text{by (1) no monopole field} \]

\[ \mathbf{A}_m(\mathbf{r}) = -\frac{1}{c} \left[ \oint_S \mathbf{v} \cdot \mathbf{J}(\mathbf{r}') \right] \frac{1}{r} = \frac{1}{c} \oint_S \mathbf{v} \cdot \mathbf{J}(\mathbf{r}') \frac{1}{r} \cdot \mathbf{v} \frac{1}{r} \]

\[ = -\frac{1}{ac} \oint_S \left[ \mathbf{J}(\mathbf{r}) \left( \mathbf{r} \cdot \mathbf{A} \right) - \mathbf{v} \left( \mathbf{A}(\mathbf{r}') \cdot \mathbf{v} \right) \right] \]

\[ -\frac{1}{ac} \oint_S \left[ \mathbf{J}(\mathbf{r}) \left( \mathbf{r} \cdot \mathbf{A} \right) + \mathbf{v} \left( \mathbf{A}(\mathbf{r}') \cdot \mathbf{v} \right) \right] \]
\[ A^2(\vec{r}) = -\frac{1}{2c^2} \oint \vec{v} \cdot (\hat{\vec{r}'} \times \hat{\vec{f}}(\vec{r}')) \times \hat{\vec{v}} \frac{1}{r} \]

where we used
\[ \vec{A} \times \vec{B} = (\vec{A} \cdot \vec{c})\vec{B} - (\vec{B} \cdot \vec{c})\vec{A} \]

BAC CAB rule

\[ \vec{m} = \frac{I}{c} \hat{\vec{r}} \]

Problem 2-21 asks you to verify that this is consistent with our prior definition of \( \vec{m} = \frac{I}{c} \hat{\vec{r}} \).

\[ \vec{A}(\vec{r}) = \vec{m} \times \hat{\vec{r}} = \frac{\vec{m} \times \hat{\vec{r}}}{r^3} \]

Vector potential of magnetic dipole

\[ \vec{r} \times \hat{\vec{f}}(\vec{r}) = \text{vector in } \vec{E} \text{ direction} \]

\[ \vec{B}^{(2)} = \vec{\nabla} \times \vec{A}^{(2)} \]

\[ = \vec{\nabla} \times \left( \frac{\vec{m} \times \hat{\vec{r}}}{r^3} \right) = \left( \frac{\vec{\nabla} \cdot \hat{\vec{r}}}{r^3} \right) \times \left( \vec{m} \times \hat{\vec{r}} \right) + \frac{1}{r^3} \vec{\nabla} \times \left( \vec{m} \times \hat{\vec{r}} \right) \]

\[ = \vec{m} \cdot \left( \left( \frac{\vec{\nabla} \cdot \hat{\vec{r}}}{r^3} \right) \vec{r} - \left( \frac{\vec{\nabla} \times \hat{\vec{r}}}{r^3} \right) \vec{r} \right) \]

\[ + \frac{1}{r^3} \left( \vec{m} \times \left( \hat{\vec{r}} \times \hat{\vec{r}} \right) - \left( \vec{m} \cdot \hat{\vec{r}} \right) \vec{r} \right) \]

\[ = -\frac{\vec{m}}{r^3} \left( 3 \frac{\vec{r} \cdot \vec{v}}{r^5} + \frac{3 \vec{r} \cdot \vec{m}}{r^5} + \frac{1}{r^3} \frac{3 \vec{m} - \frac{1}{r^3} \vec{m}}{r^3} \right) \]

\[ \vec{B}^{(2)} = \frac{1}{r^5} \left( 3 \vec{m} \cdot \hat{\vec{r}} \vec{r} - \vec{m} r^2 \right) \]

This is the same field that we got for the electric field of an electric dipole!

Again: This formula only applies far away from the current distribution. "Inside" the dipole the fields of magnetic & electric dipoles look very different.

[Diagram of electric field lines] vs. [Diagram of magnetic field lines]
Poisson & Laplace (History of the two people)

\[ \nabla^2 \Phi = \Delta \Phi = -4\pi G \] (Poisson)

\[ \nabla^2 \phi = \Delta \phi = 0 \] (Laplace)

Solutions to Laplace's eq. are called harmonic functions.

a) Definition of the problem:
   - Finite volume \( V \)
   - Boundary: Closed surface \( S \) with outward normal \( \hat{n} \)
   - No change inside \( V \) \( \Rightarrow \Delta \Phi = 0 \)
   - On boundary either \( \Phi \) is prescribed
     or "mixed" (parts of \( S \) \( \Phi \) is prescribed, other parts \( \hat{n} \cdot \nabla \Phi \) is prescribed)

b) Properties of \( \Phi \)
   1. Superposition: \( \Phi_1, \Phi_2 \) are solutions of Laplace eq.
      with different b.c. (but of the same type), then
      \( a \Phi_1 + b \Phi_2 \) is also a solution, but with suitably modified b.c.

   2. Uniqueness: \( \Phi_1, \Phi_2 \) are solutions of Laplace's eq.
      with the same b.c. Then \( \Phi_1 - \Phi_2 = \text{const} \)
      (\( \text{Const} = 0 \) unless \( \hat{n} \cdot \Phi \) is prescribed on all of \( S \))

   If we find one solution regardless of method we've done.
3. Smoothing $\Phi$ has no minima or maxima inside $\Omega$.

**Proof 1:** Laplace's equation is linear.

**Proof 2:** $\Phi - \Phi_1$ satisfies Laplace's eq. with "null b.c." either $\Phi = 0$ or on $\partial \Omega$, $n \cdot \nabla \Phi = 0$.

Look at $\iint_S da \Phi (n \cdot \nabla \Phi) = 0$

$= \iint_S da \ n \cdot (\nabla \Phi)$

$= \iiint_V n \cdot (\nabla \Phi)$

$= \iiint_V (|\nabla \Phi|^2 + \Phi \nabla^2 \Phi)$

$= \iiint_V |\nabla \Phi|^2 = 0$

Since $|\nabla \Phi|^2$ is not negative, $\nabla \Phi = 0$ & $\Phi = \text{const.}$

**Proof of 3:** If $\Phi$ has a maximum, then there exists a surface $S$ around the point where $\Phi$ has a maximum such that $n \cdot \nabla \Phi < 0$ on $S$. This means $\Phi$ decreases on $S$.

**Contradiction:** $0 > \iint_S da \ n \cdot \nabla \Phi = \iiint_V n \cdot \nabla \Phi = 0$ since $\Phi$ is no change.

So minimum $0 < \ldots \ldots = 0$

---

So how do we solve this eq?

Laplace eq. in rectangular cords

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

There are mathematical theories that explain why the strategy we take here works. As physicists all we care about is that we found the solution.

We will take a pragmatic approach:

a) First find a general solution irrespective of boundary condition

b) Match the solution to the b.c.
Separation of Variables

Look for a form \( \Phi = X(x)Y(y)Z(z) \)

\[ \Delta \Phi = \nabla^2 \Phi = \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} + X Y \frac{\partial^2 Z}{\partial z^2} = 0 \]

Divide by \( \Phi = X Y Z \)

\[ \frac{1}{X} \frac{\partial^2 X}{\partial x^2} + \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = 0 \]

Each of these terms must be a constant since the functions are independent.

\[ \frac{1}{X} \frac{\partial^2 X}{\partial x^2} = \lambda^2 \quad \text{or} \quad \frac{\partial^2 X}{\partial x^2} = \lambda^2 X \]

where \( \lambda^2 + \beta^2 + \gamma^2 \neq 0 \)

This means \( \lambda, \beta, \gamma \) can't all be real. \#5

\[ \frac{\partial^2 Y}{\partial y^2} = \beta^2 Y \]

\[ \frac{\partial^2 Z}{\partial z^2} = \gamma^2 Z \]

One or two must be imaginary.

Solutions are exponential functions

\[ X(x) = e^{\alpha x} \quad \text{or} \quad (\cosh \alpha x, \sinh \alpha x, e^{-\alpha x}, \text{etc.}) \]

\[ Y(y) = e^{\beta y} \]

\[ Z(z) = e^{\gamma z} \]

General solution is:

\[ \Phi(x,y,z) = \sum A_{\alpha \beta \gamma} e^{\alpha x + \beta y + \gamma z} \]

Ex:

Square metal pipe with walls held at \( \Phi = 0 \)

Capped by a plate at \( x = 0 \) held at \( \Phi = \Phi_0 \)

Note: Plate is separated by a small gap

What is \( \Phi \) inside the pipe?
Deal w/ boundary conditions first.

Choose \( \bar{\Omega}(r) \) such that \( \bar{\Omega}(a) = \bar{\Omega}(a) = 0 \)

\[
\bar{\Omega}(r) = \sin \frac{\pi r a}{a}
\]

Choose \( \bar{\zeta}(z) \) such that \( \bar{\zeta}(0) = \bar{\zeta}(a) = 0 \)

\[
\bar{\zeta}(z) = \sin \frac{\pi z a}{a}
\]

\( X(x) \) must go to 0 as \( x \to \infty \)

\[
X(x) = e^{-\alpha x}
\]

\[
\left( \frac{\pi r a}{a} \right)^2 + \left( \frac{\pi z a}{a} \right)^2 + \alpha^2 = 0 \quad \alpha_{rs} = \frac{\pi}{a} \sqrt{r^2 + s^2}
\]

\[
\bar{\Omega}(x, y, z) = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \alpha_{rs} e^{-\alpha x} \sin \frac{\pi r a}{a} \sin \frac{\pi z a}{a}
\]

What about boundary conditions at \( x = 0 \)

\[
\bar{\Omega}(0, y, z) = \bar{\Omega}_0 = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \alpha_{rs} \bar{f}_{rs}(y, z)
\]

The functions \( \bar{f}_{rs}(y, z) \) are complete and orthogonal on \( 0 \leq y, z \leq a \)

Completeness: Any function \( F(y, z) \) can be written as a linear combination of the \( \bar{f}_{rs} \) functions

Orthogonality: \[
\int_0^a \int_0^a \bar{f}_{rs}(y, z) \bar{f}_{r's'}(y, z) \, dy \, dz = \delta_{rr'} \delta_{ss'} \alpha_{rs}
\]

\[
\alpha_{rs} = \frac{a^2}{4} \text{ for our functions } \bar{f}_{rs}
\]

This is not a coincidence. This is a general property of solutions of differential equations of the type.
we considered, with null boundary conditions
"Sturm-Liouville Problem"
Completeness guarantees that suitable coefficients exist
\[ \Phi_0 = \sum_{r,s=1}^{\infty} a_r s_r(y,z) = \Phi(0,y,z) \]
Orthogonality gives a quick way to find them
\[ \int_0^a dydz \Phi(0,y,z) s_{r,s}(y,z) = \sum_{r,s=1}^{\infty} \int_0^a dydz a_r s_{r,s}(y,z) s_{r,s}(y,z) = \]
\[ = a_r s_{r,s}^2 = a_r^2 \]
but we also have
\[ \int_0^a dydz \Phi(0,y,z) s_{r,s}(y,z) = \int_0^a dydz \Phi_0 \sin \frac{\pi yr}{a} \sin \frac{\pi zr}{a} \]
\[ = \begin{cases} \frac{2a}{\pi r} \frac{2a}{\pi s} & \text{if } r', s' \text{ odd} \\ 0 & \text{otherwise} \end{cases} \]
\[ a_r = \begin{cases} \frac{16 \Phi_0}{\pi^2 rs} & \text{if } r', s' \text{ odd} \\ 0 & \text{otherwise} \end{cases} \]
\[ \Phi(x,y,z) = \sum_{r,s=1}^{\infty} \frac{16 \Phi_0}{\pi^2 rs} e^{-\frac{\pi x}{a} \sqrt{r^2 + s^2}} \sin \frac{\pi yr}{a} \sin \frac{\pi zr}{a} \]
for \( x \gg a \) \( \Phi \) dominated by \( r=s=1 \) term
\[ \Phi(x,y,z) \approx \frac{16 \Phi_0}{\pi^2} \sin \frac{\pi yr}{a} \sin \frac{\pi zr}{a} e^{-\frac{\pi x}{a} \sqrt{1}} \]
Again our Strategy

* Find Special Solutions using separation of variables
* General solution will be a superposition of the special solutions
* Match solution to boundary condition
* Use completeness & orthogonality of special solutions

For rectangular co-ords: 
\[ \Phi(x, y, z) = \sum \lambda_{\alpha\beta\gamma} e^{\alpha x + \beta y + \gamma z} \]

\[ \alpha, \beta, \gamma \geq 0 \]
\[ \alpha, \beta, \gamma \text{ one or two real & the other imaginary} \]

Laplace Eq for Spherical Coords r, \( \theta \), \( \phi \)

a) \[ \nabla^2 \Phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} = 0 \]

b) Separation of variables
\[ \Phi(r, \theta, \phi) = R(r) P(\theta) Q(\phi) \]
\[ \frac{1}{\Phi} \nabla^2 \Phi = 0 \]

\[ \frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{1}{P \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dP}{d\theta} \right) + \frac{1}{Q \sin^2 \theta} \frac{d^2Q}{d\phi^2} = 0 \]

c) Multiply by \( r^2 \sin^2 \theta \)
\[ \frac{\sin^2 \theta}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{\sin \theta}{P} \frac{d}{d\theta} \left( \sin \theta \frac{dP}{d\theta} \right) + \frac{1}{Q \sin^2 \theta} \frac{d^2Q}{d\phi^2} = 0 \]

\[ \frac{1}{Q} \frac{d^2Q}{d\phi^2} = \text{constant} = -m^2 \]

\[ Q(\phi) = e^{i m \phi} \text{ since } Q(\phi) = Q(\phi + 2\pi) \]

\[ m = 0, \pm 1, \pm 2 \]
d) Return to (b) but now divide by \( r^2 \)

\[
\frac{1}{R} \frac{d}{dr} \left( R^2 \frac{dr}{dr} \right) + \frac{1}{\rho \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d \rho}{d\theta} \right) \frac{m^2}{\sin \theta} = 0
\]

\[\text{constant} \quad \ell (l+1) \]

\[\text{constant} \quad \ell (l+1) = -\ell (l+1) \]

\[
\frac{d}{dr} \left( R^2 \frac{dr}{dr} \right) - \ell (l+1) R = 0
\]

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \sin \theta \frac{d \rho}{d\theta} - \frac{m^2 \rho}{\sin \theta} + \ell (l+1) \rho = 0
\]

e) Solve for \( R \):

we try \( R(r) = r^x \) \( \alpha = l \alpha - l - 1 \)

\[
R_e(r) = A e^l + B e^{-l-1}
\]

Note: if origin is part of the volume where \( \nabla \Phi \) is being solved then \( B = 0 \) otherwise \( A e^l \) or \( B e^{-l-1} \) may represent

f) Solve eq for \( \rho \)

Make variable change \( \rho \Phi \to x \)

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \sin \theta \frac{d \rho}{d\theta} \Rightarrow \frac{d}{dx}
\]

\[\frac{d}{dx} \left( \left( 1-x^2 \right) \frac{d \rho}{dx} \right) + \left[ \ell (l+1) - \frac{m^2}{1-x^2} \right] \rho = 0
\]

First lets solve for \( m = 0 \) axial symmetry no dependence on \( \theta \)

Solutions are the Legendre polynomials

Then we will discuss the case for arbitrary \( m \)

Solutions are the Spherical Harmonics
The Legendre Polynomial

a) Rewrite equation for \( P \)
\[
(1-x^2)\frac{d^2}{dx^2}P - 2x \frac{d}{dx}P + l(l+1)P = 0
\]
Legendre's Eq.

Method for solving this eq.

- **Ansatz:** \( P \) is a power series in \( x \)
- **Require Convergence for** \(-1 \leq x \leq 1\)
  
  we will find that convergence occurs only if
  
  power series breaks off & \( P \) becomes a polynomial in \( x \)
- **Degree of polynomial will be** \( l \)

"Legendre polynomial of order \( l \)" \( P_l(x) \)

**Ansatz:** \( P(x) = \sum_{n=0}^{\infty} a_n x^n \)

plug into eq.

\[
(1-x^2)\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - 2x \sum_{n=0}^{\infty} n a_n x^{n-1} + l(l+1)\sum_{n=0}^{\infty} a_n x^n = 0
\]

Collect all equal powers & solve for the coefficients \( a_n \)

\[
a_n \sum_{n=0}^{\infty} l(l+1) - n(n-1) - 2 n^2 + (n+2)(n+1) a_{n+2} = 0
\]

\[
a_{n+2} = \frac{a_n (l(l+1) - n(n+1))}{(n+1)(n+2)}
\]

Generally \( \frac{a_{n+2}}{a_n} \to -1 \) if \( n \to \infty \)

**Note:** This power series does not converge for \( x = \pm 1 \)

Unless \( l \) is an integer, then series terminates when \( n = l \) since \( a_n = 0 \) for all \( n > l \)

Solution w/ integer \( l \) is denoted \( P_l(x) \) & is a polynomial

of degree \( l \)
b) Properties of Legendre Polynomials

1) If \( l \) is even all \( \text{Legendre} \) \( n \) odd or zero:
\[
P(-x) = (-1)^l P(x)
\]

2) To normalize for all order \( P_0(1) = 1 \)

3) Complete orthogonal set on \(-1 \leq x \leq 1\):
\[
\int_{-1}^{1} dx P_l(x) P_m(x) = \int_{-1}^{1} \frac{dx}{2l+1}
\]

4) \( P_0(x) = 1 \)
\( P_1(x) = x \)
\( P_2(x) = \frac{1}{2} (3x^2 - 1) \)
\( P_3(x) = \frac{1}{2} (5x^3 - 3x) \)
\( P_4(x) = \frac{1}{8} (35x^4 - 30x^2 + 3) \)
\( P_5(x) = \frac{1}{8} (63x^5 - 70x^3 + 15x) \)

5) Rodrigues' formula
\[
P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l
\]

6) Generating function
\[
F(x, \mu) = \frac{1}{(1 - 2x \mu + \mu^2)^{1/2}} = \sum_{l=0}^{\infty} \mu^l P_l(x)
\]

IE: expanding as a power series in \( \mu \), coefficients are polynomials in \( \mu \)
Recursion relations

\[(l+1) P_{l+1}(x) = (2l+1) x P_l(x) - l P_l(x)\]

\[(1-x^2) \frac{d P_l}{d x} = -lx P_l(x) + l P_{l-1}(x)\]

Summary: General solution of Laplace's eq w/ axial symmetry

\[\Phi(r, \theta) = \sum_{l=0}^{\infty} \left( A_l r^l + B_l \frac{1}{r^{l+1}} \right) P_l(\cos \theta)\]

Fix \(A_l, B_l\) using boundary conditions

Note: if \(\Phi\) is continuous at \(r=0\) \(B_l=0\)

For \(\Phi\) corresponding to localized source \(A_l=0\)

Note: Similar to multipole expansion. Same \(l\) dependence

For given order in expansion \(B_0=\) monopole, \(B_1=\) dipole, etc.

Spherical Harmonics \(m \neq 0\)

\[\frac{d}{dx} \left( (1-x^2) \frac{d P_l}{d x} \right) + (l(l+1) - \frac{m^2}{1-x^2}) P_l = 0\]

Solutions are called "associated Legendre functions"

\[P_l^m(x) = (-1)^m \frac{(1-x^2)^{\frac{m}{2}}}{2^l l!} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l, \quad m=0, \pm 1, \pm 2, \ldots \pm l\]

* Orthogonality: \[\int_{-1}^{1} dx P_l^m(x) P_l^{m'}(x) = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{mm'}\]

* Together with \(P_l^0\), the associated Legendre functions are complete on \(0 \leq \theta \leq \pi, \ 0 \leq \phi \leq 2\pi\)

Spherical Harmonics:

\[Y_l^m(\theta, \phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos \theta) e^{im\phi}\]
\[ \int_0^{2\pi} \int_0^\pi Y_{lm}^* (\theta, \phi) \, r^2 \sin \theta \, dr \, d\theta = \delta_{lm} \delta_{m^*} \]

\[ Y_0^0 (\theta, \phi) = \frac{1}{\sqrt{4\pi}} \]

\[ Y_1^0 (\theta, \phi) = \frac{\sqrt{3}}{\sqrt{4\pi}} \cos \theta \]

\[ Y_{\pm 1}^1 (\theta, \phi) = \frac{\sqrt{3}}{2\sqrt{4\pi}} \sin \theta \exp \pm i\phi \]

\[ \Rightarrow \Psi (r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (A_e^m r^l + B_e^m \frac{1}{r^{l+1}}) Y_{lm} (\theta, \phi) \]

**Review & Examples**

1) Laplace's equation for spherical polar coordinates \((r, \theta, \phi)\)

\[ \Delta \Phi = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right) + \frac{1}{r^2 \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Phi}{d\theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{d^2 \Phi}{d\phi^2} \]

a) Find a special solution of the form

\[ \Phi (r, \theta, \phi) = R(r) \, P(\cos \phi) \, Q(\theta) \]

\[ Q_m (\theta) = \exp \pm im \phi \quad m = 0, \pm 1, \pm 2, \ldots \]

\[ R_l (r) = r^l \cos \frac{1}{r^{l+1}} \quad l = m, m+1, \ldots \]

\[ P \text{ satisfies} \quad (1-x^2) \frac{d^2 P}{dx^2} + (l(l+1) - \frac{m^2}{1-x^2}) P = 0 \]

If \( m = 0 \)

\[ P = P_l (\cos \phi) \quad l^{th} \text{ Legendre polynomial} \]

\[ P_l (x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l \]
b) General Solution for $m=0$ Case (no $\phi$ dependence)

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

- $P_l$ are complete & orthogonal on $-1 \leq \cos \theta \leq 1$

$$\int_{-1}^{1} P_l(\cos \theta) P_m(\cos \theta) \, d\cos \theta = \delta_{lm} \frac{2}{2l+1}$$

- $P_0 \cos \theta = 1$
- $P_1 \sin \theta = \cos \theta$
- $P_2 \cos \theta = \frac{1}{2} (3 \cos^2 \theta - 1)$

Note similarity to multipole expansion. Some angular dependence for a given order in the expansion.

$A_l$ are zero for a localized source.

$C_l$ is $m \neq 0$, we need spherical harmonics.

Examples:
1) Conducting sphere of radius $a$ in uniform electric field

Sphere alters the field locally but far from sphere we expect no change. Find $\vec{E}$ field everywhere.

$$\vec{E} = \vec{E}_0 \hat{r}_2$$

- Axial Symmetry means $m=0$

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

b) For $r \to \infty$ $\vec{E} = E_0 \hat{r}$  $\Phi = -E_0 \hat{r}$  $=-E_0 r \cos \theta$
Comparing w/ expansions

\[ A_l = 0 \text{ if } l \neq 1 \quad A_1 = -E_0 \]

So \[ \Phi(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta) - E_0 r \cos \theta \]

(c) Match \( B \) using boundary conditions at sphere.

\( \Phi \) is constant at \( r = a \) let make it \( \Phi = 0 \) (far E-field outside)

use orthogonality of the \( P_l \) to find \( B_l \)

\[ \Phi(a, \theta) = -E_0 a P_1 + \sum_{l=0}^{\infty} \frac{B_l}{a^{l+1}} P_l = 0 \]

\[ E_0 a \int_0^1 P_1(x) P_l(x) dx = \sum_{l=0}^{\infty} \frac{B_l}{a^{l+1}} \int_0^1 P_l P_l(x) dx \]

\[ E_0 a \frac{2}{2l+1} \delta_{l1} = \frac{B_l}{a^{l+1}} \frac{2}{2l+1} \]

\[ B_l = E_0 a (l+2) \delta_{l1} \quad B_1 = E_0 a^3 \quad B_{l+1} = 0 \]

\[ \Phi(r, \theta) = -E_0 (1 - \frac{a^3}{r^3}) r \cos \theta \]

(d) **Short Cut**

Field is \( \perp \) to sphere since sphere is a conductor.

Changes form a dipole — guess that only \( l = 1 \) contributes.

match \( B \), using b.c. Verify b.c. are met small \( \theta \)

\& invoke uniqueness to ensure you have the solution.

e) Electric field inside sphere \( \vec{E} = 0 \)

outside \[ \begin{cases} E_r = -\frac{1}{r} \frac{\partial \Phi}{\partial r} = E_0 \left( 1 + \frac{2a^3}{r^3} \right) \cos \theta \\ E_\theta = -\frac{1}{r} \frac{\partial \Phi}{\partial \theta} = -E_0 \left( 1 - \frac{a^3}{r^3} \right) \sin \theta \end{cases} \]

note as \( r = a \)

\[ E_0 = 0 \]
Surface charge density is \( E_r(r=a) = 4\pi \sigma \).

\[ \text{used} \ (E_2 E_2 - E_1 E_1) \cdot \mathbf{n} = 4\pi \sigma, \quad \sigma = \frac{3}{4\pi} E_0 \rho_\text{ind} \]

So the field inside sphere is zero \( \rho = E_2 = 1 \)

\[
\Phi = -E_0 r \cos \theta + \frac{E_0 a^3 \rho_\text{ind}}{r^2} \quad r > a
\]

\[ \text{External Contribution} \]
\[ \text{Local induced Contribution} \]

\( \Phi \) induced is just the dipole potential with \( P = E_0 a^3 \).

**Ex 2** Let's repeat this for a dielectric sphere

a) Axial symmetry \( \Rightarrow \Phi(r, \theta) = \frac{2}{\varepsilon_0} (\alpha r^2 + \beta \frac{1}{r^2+1}) \rho_\text{ind}(r, \theta) \)

b) for \( r \to a \) \( E = E_0 e_z \Rightarrow \Phi = -E_0 e_z = -E_0 r \cos \theta \)

c) Guess: need \( l = 1 \) only

\( r > a \): \( \Phi(r, \theta) = -E_0 r \cos \theta + \frac{B_1 \cos \theta}{r^2} \)

This time dipole will not entirely cancel field inside.

\( r < a \): \( \Phi(r, \theta) = A_1 r \cos \theta \)

Here we have assumed that the applied field does not lead to a charge density inside the sphere (i.e. polarization of sphere is uniform). Otherwise \( \Phi \) does not obey the Laplace eq. for \( r < a \)

b.c. at \( r = a \)

- \( D_z = D_r \) continuous
- \( E_z = E_x \) continuous
- \( \Phi \) continuous

\( \Phi \) at \( a \)

\[-E_0 a \cos \theta + B_1 \cos \theta = A_1 a \cos \theta \]

\( \Phi \) continuous end-vec. \( E_z \), continuous.