b) A Couple of Math results.

① $\int_V dv' \vec{j}(\vec{r}') = 0$

② $\int_V dv' \{ \vec{j}(\vec{r}')(\vec{r}'\cdot\vec{k}) + \vec{r}'(\vec{j}(\vec{r}')\cdot\vec{k}) \} = 0$ for any $\vec{k}$

Use a "physicists proof"

write $\vec{j}(\vec{r}) = \sum_\alpha q_\alpha \delta(\vec{r}-\vec{r}_\alpha)\dot{\vec{r}}_\alpha$

① $\int_V dv' \vec{j}(\vec{r}') = \sum_\alpha q_\alpha \dot{\vec{r}}_\alpha = \frac{d}{dt}\left(\sum_\alpha q_\alpha \vec{r}_\alpha\right) = 0$ for stationary currents

currents have no end or begining so an integral over a current loop gives zero. just as much current flowing in one direction as any other.

② $\int_V dv' \{ \quad \} = \sum_\alpha q_\alpha\left(\dot{\vec{r}}_\alpha(\vec{r}_\alpha\cdot\vec{k}) + \vec{r}_\alpha(\dot{\vec{r}}_\alpha\cdot\vec{k})\right)$

$= \frac{d}{dt}\sum_\alpha q_\alpha\left(\vec{r}_\alpha\underbrace{(\vec{r}_\alpha\cdot\vec{k})}_{\text{const}}\right) = 0$ for stationary currents

Same reasoning integral over a current loop gives zero.

c) $\vec{A}(\vec{r}) = \vec{A}^{(1)}(\vec{r}) + A^{(2)}(\vec{r}) + \dots$

$\vec{A}^{(1)}(\vec{r}) = \frac{1}{c}\left[\int_V dv' \vec{j}(\vec{r}')\right]\frac{1}{r} = 0$ by ① no monopole field

$A^{(2)}(r) = -\frac{1}{c}\left[\int_V dv' \vec{j}(\vec{r}')\,\vec{r}'\right]\cdot\vec{\nabla}\frac{1}{r}$ original terms

$= -\frac{1}{2c}\int_V dv'\left[\vec{j}(\vec{r}')(\vec{r}'\cdot\vec{\nabla}\frac{1}{r}) - \vec{r}'(\vec{j}(\vec{r}')\cdot\vec{\nabla}\frac{1}{r})\right]$ These two cancell

$-\frac{1}{2c}\int_V dv'\left[\vec{j}(\vec{r}')\underbrace{(\vec{r}'\cdot\vec{\nabla}\frac{1}{r}) + \vec{r}'(\vec{j}(\vec{r}')\cdot\vec{\nabla}\frac{1}{r})}_{=0 \text{ by } ②}\right]$

$$A^{2}(\vec{r}) = -\frac{1}{2c}\oint_V dv'(\vec{r}'\times\vec{j}(\vec{r}'))\times\vec{\nabla}\frac{1}{r}$$

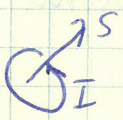
where we used $\vec{A}\times\vec{B}\times\vec{C} = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$

BAC CAB rule

$$\equiv -\vec{m}\times\vec{\nabla}\frac{1}{r}$$

with $\vec{m} = \frac{1}{2c}\oint_V dv'(\vec{r}'\times\vec{j}(\vec{r}'))$ the magnetic dipole moment

Problem 2-21 asks you to verify that this is consistent w/ our prior definition of $\vec{m} = \frac{I}{c}\vec{S}$

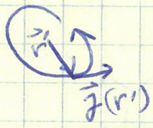


$$\vec{m} = \frac{I}{c}\vec{S}$$

$$\vec{A}^{(2)}(\vec{r}) = \frac{\vec{m}\times\hat{e}_r}{r^2} = \frac{\vec{m}\times\vec{r}}{r^3}$$

$\vec{r}'\times\vec{j}(\vec{r}')$ = vector in $\perp$ direction

Vector potential of magnetic dipole



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d) $$\vec{B}^{(2)} = \vec{\nabla}\times\vec{A}^{(2)}$$

used $\vec{\nabla}\times f\vec{A} = f\,\vec{\nabla}\times\vec{A} - \vec{A}\times\vec{\nabla}f$

$$= \nabla\times\left(\frac{\vec{m}\times\vec{r}}{r^3}\right) = \left(\vec{\nabla}\frac{1}{r^3}\right)\times(\vec{m}\times\vec{r}) + \frac{1}{r^3}\vec{\nabla}\times(\vec{m}\times\vec{r})$$

$$= \vec{m}\cdot\left(\left(\vec{\nabla}\frac{1}{r^3}\right)\cdot\vec{r}\right) - \left(\left(\vec{\nabla}\frac{1}{r^3}\right)\cdot\vec{m}\right)\vec{r}$$

BAC-CAB rule ~~twice~~

$$+ \frac{1}{r^3}\left(m(\vec{\nabla}\cdot\vec{r})\right) - \left(\vec{m}\cdot\vec{\nabla}\right)\vec{r}$$

$\vec{\nabla}\times\vec{A}\times\vec{B} = (B\cdot\nabla)A - (A\cdot\nabla)B + A(\nabla\cdot B) - B(\nabla\cdot A)$

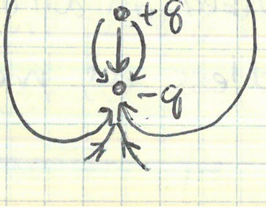
$$= \cancel{-\vec{m}\,\frac{3\vec{r}\cdot\vec{r}}{r^5}} + \frac{3\,\vec{r}\cdot\vec{m}}{r^5}\vec{r} + \cancel{\frac{1}{r^3}3\vec{m}} - \frac{1}{r^3}\vec{m}$$

magnetic moment $\vec{m}$ is constant w/ respect to curls derivatives which operate on $\vec{r}$ so $(\vec{r}\cdot\vec{\nabla})\vec{m}$ & $r(\vec{\nabla}\cdot\vec{m})$ term are zero.

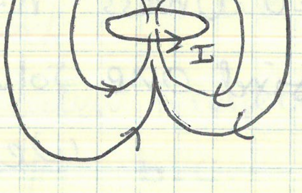
$$\vec{B}^{(2)} = \frac{1}{r^5}\left(3(\vec{m}\cdot\vec{r})\vec{r} - \vec{m}r^2\right)$$

This is the same field that we got for the electric field of an electric dipole!

Again: This formula only applies far away from the current charge distribution. "Inside" the dipole the fields of magnetic & electric dipoles look very different



vs.



Poisson & Laplace (History of the two people)

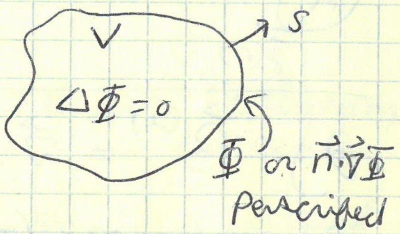
$$\nabla^2 \Phi = \Delta \Phi = -4\pi\rho \quad \text{(Poisson)}$$

$$\nabla^2 \Phi = \Delta \Phi = 0 \quad \text{(Laplace)}$$

who when asked about why his Books on Celestial mechanics did not mention "God" answered: "I had no need of that Hypothesis"

Solutions to Laplace's eq are called "harmonic functions"

a) Definition of the problem:



- finite Volume V
- boundary: closed Surface S w/ outward normal $\hat{n}$
- No Charge inside $V \Rightarrow \Delta\Phi = 0$
- On boundary either Φ is perscribed or "mixed" (parts of S Φ is perscribed other parts $\hat{n}\cdot\vec{\nabla}\Phi$ is perscribed)

b) Properties of Φ

① Superposition: Φ_1, Φ_2 are Solutions of Laplace eq w/ different b.c. (but of the same type) then $a\Phi_1 + b\Phi_2$ is also a solution, but with suitably modified b.c.

② Uniqueness: Φ_1, Φ_2 are Solutions of Laplace's eq with the same b.c. Then $\Phi_1 - \Phi_2 = \text{Const}$ (Const $= 0$ unless $\hat{n}\cdot\vec{\nabla}\Phi$ is perscribed on all of S)

$\therefore$ If we find one solution regardless of method were done.

③ Smoothing Φ has no minima or maxima inside V

proof of ① Laplace's equation is linear

proof of ② $\Phi = \Phi_1 - \Phi_2$ satisfies Laplace's eq with "null b.c." either $\Phi = 0$ on S or $\vec{\nabla}\Phi \cdot \hat{n} = 0$

look at $\oint_S da\, \Phi (n \cdot \vec{\nabla}\Phi) = 0$

$$= \oint_S da\, n \cdot (\Phi \nabla \Phi)$$

$$= \int_V dv\, \vec{\nabla} \cdot (\Phi \vec{\nabla}\Phi)$$

$$= \int_V dv\, (|\vec{\nabla}\Phi|^2 + \Phi \underbrace{\nabla^2\Phi}_{=0})$$

$$= \int_V dv\, |\vec{\nabla}\Phi|^2 = 0$$

Since $|\vec{\nabla}\Phi|^2$ is not negative $\vec{\nabla}\Phi = 0$ & Φ = Const.

Proof of ③ if Φ has a maximum then there exists a surface S around the point where Φ has a maximum such that $\hat{n} \cdot \vec{\nabla}\Phi < 0$ on S. This means

Contradiction $0 > \oint_S da\, \hat{n} \cdot \vec{\nabla}\Phi = \int_{V_S} dv\, \nabla^2 \Phi = 0$ since there is no charge

for minimum $0 < \dots = 0$

So how do we solve this eq?

Laplace eq. in rectangular coords

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

There are mathematical theories that explain why the strategy we take here works. as physicists all we care about is that we found the solution

We will take a pragmatic approach:

a) First find a general solution irrespective of boundary conditions

b) Match the solution to the b.c.

Seperation of Variables

Look for Φ of the form $\Phi = X(x)\,Y(y)\,Z_1(z)$

$$\Delta\Phi = Y Z_1 \frac{\partial^2 X}{\partial x^2} + X Z_1 \frac{\partial^2 Y}{\partial y^2} + X Y \frac{\partial^2 Z_1}{\partial z^2} = 0$$

divide by $\Phi = XYZ_1$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z_1}\frac{\partial^2 Z_1}{\partial z^2} = 0$$

each of these terms must be a constant since the functions are independent

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \alpha^2 \quad \text{or} \quad \frac{\partial^2 X}{\partial x^2} = \alpha^2 X$$

$$\frac{\partial^2 Y}{\partial y^2} = \beta^2 Y$$

$$\frac{\partial^2 Z_1}{\partial z^2} = \gamma^2 Z_1$$

where $\alpha^2+\beta^2+\gamma^2=0$

this means α, β, γ can't all be real #'s

one or two must be imaginary

Solutions are exponential functions

$X(x) = e^{\alpha x}$ or ($\cosh \alpha x$, $\sinh \alpha x$ $e^{-\alpha x}$ etc.)

$Y(y) = e^{\beta y}$ "

$Z_1(z) = e^{\gamma z}$ "

if $\alpha = i\alpha'$

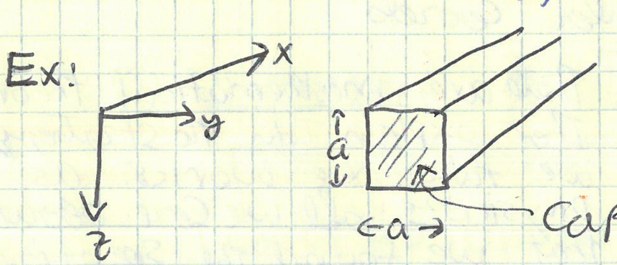
then $e^{i\alpha' x}$ or $e^{-i\alpha' x}$ or

$\sin\alpha' x$ or $\cos\alpha' x$

General Solution is:

$$\Phi(x,y,z) = \sum_{\alpha,\beta,\gamma} A_{\alpha\beta\gamma}\, e^{\alpha x + \beta y + \gamma z}$$

Ex:



square metal pipe

with walls held at $\Phi = 0$

capped by a plate at $x=0$ held at $\Phi = \Phi_0$

note: Plate is separated by a small gap

what is Φ inside the pipe?

Deal w/ boundary conditions first.

Choose $Y(y)$ such that $Y(0) = Y(a) = 0$

$$Y_r(y) = \sin \frac{\pi y r}{a} \qquad r = 1, 2, \cdots$$

Choose $Z_1(z)$ such that $'Z_1(0) = Z_1(a) = 0$

$$Z_{1_s}(z) = \sin \frac{\pi z s}{a} \qquad s = 1, 2, \cdots$$

$X(x)$ must go to 0 if $x \to \infty$

$$X_\alpha(x) = e^{-\alpha x}$$

$$\left(\frac{\pi r}{a}\right)^2 + \left(\frac{\pi s}{a}\right)^2 + \alpha^2 = 0 \qquad \alpha_{rs} = \frac{\pi}{a}\sqrt{r^2 + s^2}$$

$$\Phi(x, y, z) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} A_{rs} e^{-\alpha_{rs} x} \sin \frac{\pi y r}{a} \sin \frac{\pi z s}{a}$$

What about boundary conditions at $x = 0$

$$\Phi(0, y, z) = \Phi_0 = \sum_{r,s=1}^{\infty} A_{rs} f_{r,s}(y, z) \qquad f_{r,s}(y,z) = \sin \frac{\pi y r}{a} \sin \frac{\pi z s}{a}$$

The functions $f_{rs}(y, z)$ are complete & orthogonal on

$0 < y, z < a$

Completeness: any function $F(y, z)$ can be written as a linear combination of the f_{rs} functions

Orthogonality: $\int_0^a dy\,dz\, f_{r,s}(y, z)\, f_{r's'}(y, z) = \delta_{rr'}\delta_{ss'} C_{rs}$

$$C_{rs} = \frac{a^2}{4} \text{ for our functions } f_{rs}$$

This is not a coincidence. This is a general property of solutions of differential equations of the type we Considered, with null boundary Conditions "Sturm - Liouvill problem"

Completeness guarantees that Suitable Coefficients A_{rs} exist

$$\Phi_0 = \sum_{r,s=1}^{\infty} A_{rs} f_{rs}(y,z) = \Phi(0,y,z)$$

Orthogonality gives a quick way to find them

$$\int_0^a dy\,dz\, \Phi(0,y,z)\, f_{r's'}(y,z) = \sum_{r,s=1}^{\infty} \int_0^a dy\,dz\, A_{rs} f_{rs}(y,z) f_{r's'}(y,z)$$

$$= A_{r's'} C_{r's'}$$

$$= A_{r's'} \frac{a^2}{4}$$

but we also have

$$\int_0^a dy\,dz\, \Phi(0,y,z)\, f_{r's'}(y,z) = \int_0^a dy\,dz\, \Phi_0 \sin\frac{\pi y r'}{a} \sin\frac{\pi z s'}{a}$$

$$= \Phi_0 \begin{cases} \dfrac{2a}{\pi r'} \dfrac{2a}{\pi s'} & \text{if } r', s' \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

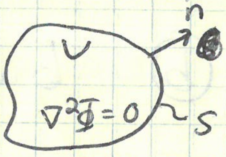
$$A_{rs} = \begin{cases} \dfrac{16}{\pi^2 rs}\Phi_0 & \text{if } r, s \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$ since $A_{rs}\frac{a^2}{4}$ has to equal

$$\Phi(x,y,z) = \sum_{\substack{r,s=1 \\ r,s \text{ odd}}}^{\infty} \frac{16\Phi_0}{\pi^2 rs} e^{-\frac{\pi x}{a}\sqrt{r^2+s^2}} \sin\frac{\pi y r}{a} \sin\frac{\pi z s}{a}$$

for $x \gg a$ Φ dominated by $r = s = 1$ term

$$\Phi(x,y,z) \approx \frac{16}{\pi^2} \sin\frac{\pi y}{a} \sin\frac{\pi z}{a} e^{-\frac{\pi x}{a}\sqrt{2}}$$

Again our strategy



b.c: on each part of S specify Φ or $\hat{n}\cdot\vec{\nabla}\Phi$

* Find Special Solutions using Separation of Variables

* general Solution will be a Superposition of the special Solutions

* Match Solution to boundary Condition use Completeness & orthogonality of special Solutions

For rectangular coords: $\Phi(x,y,z) = \sum_{\alpha,\beta,\gamma} A_{\alpha\beta\gamma}\, e^{\alpha x+\beta y+\gamma z}$

$\alpha^2+\beta^2+\gamma^2=0$

α,β,γ one or two real the other imaginary

Laplace Eq for Spherical Coords r,θ,ϕ

a) $\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial \Phi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Phi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$

b) Separation of variables

$\Phi(r,\theta,\phi) = R(r)P(\theta)Q(\phi)$ $\qquad \frac{1}{\Phi}\nabla^2\Phi = 0$

Can switch $\frac{\partial}{\partial r}\to\frac{d}{dr}$ since variables are separated

$$\frac{1}{r^2 R}\frac{d}{dr} r^2\frac{dR}{dr} + \frac{1}{r^2 P\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dP}{d\theta} + \frac{1}{r^2 Q\sin^2\theta}\frac{d^2Q}{d\phi^2} = 0$$

c) multiply by $r^2\sin^2\theta$

$$\underbrace{\frac{\sin^2\theta}{R}\frac{d}{dr}r^2\frac{dR}{dr} + \frac{\sin\theta}{P}\frac{d}{d\theta}\sin\theta\frac{dP}{d\theta}}_{\text{function of } r,\theta} + \underbrace{\frac{1}{Q}\frac{d^2Q}{d\phi^2}}_{\text{function of } \phi} = 0$$

$\frac{1}{Q}\frac{d^2Q}{d\phi^2} = \text{Const} = -m^2$

$Q(\phi) = e^{im\phi}$ since $Q(\phi) = Q(\phi+2\pi)$ $\quad m = 0, \pm1, \pm2$

d) Return to (b) but now ~~divide~~ multiply by r^2

$$\underbrace{\frac{1}{R}\frac{d}{dr}r^2\frac{dR}{dr}}_{\substack{\text{Constant}\\ \ell(\ell+1)}} + \underbrace{\frac{1}{P\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dP}{d\theta} - \frac{m^2}{\sin^2\theta}}_{\substack{\text{Constant}\\ =-\ell(\ell+1)}} = 0$$

notation $\ell(\ell+1)$ is for future reference

$$\frac{d}{dr}r^2\frac{dR}{dr} - \ell(\ell+1)R = 0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dP}{d\theta} - \frac{m^2 P}{\sin^2\theta} + \ell(\ell+1)P = 0$$

e) Solve for R: we try $R(r) = r^\alpha$ $\alpha = \ell$ or $-\ell-1$

$$R_\ell(r) = A_\ell r^\ell + B_\ell r^{-\ell-1}$$

note: if origin is part of the volume where $\nabla^2\Phi$ is being solved then $B_\ell = 0$ otherwise A_ℓ & B_ℓ may be present

f) Solve eq for P

Make variable change $\cos\theta \to x$

$$-\frac{1}{\sin\theta}\frac{d}{d\theta} \Rightarrow \frac{d}{dx}$$

$$\frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right) + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P = 0$$

- First lets solve for $m=0$ axial symmetry no dependence on ϕ
  Solutions are the Legendre Polynomials

- Then we will discuss the case for arbitrary m
  Solutions are the Spherical Harmonics

The Legendre Polynomial

a) Rewrite equation for P as

$$(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \ell(\ell+1)P = 0 \qquad \text{Legendre's Eq.}$$

method for solving this eq.

- Ansatz: P is a power series in x
- Require Convergence for $-1 \leq x \leq 1$

  we will find that Convergence occurs only if power series breaks off & P becomes a polynomial in x
- Degree of polynomial will be ℓ

"Legendre polynomial of order ℓ" $P_\ell(\cos\theta)$

Ansatz: $P(x) = \sum_{n=0}^{\infty} a_n x^n$ plug into eq.

$$(1-x^2)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=0}^{\infty} n a_n x^{n-1} + \ell(\ell+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

Collect all equal powers & Solve for the Coefficients of x^n

$$a_n\{\ell(\ell+1) - n(n-1) - 2n\} + (n+2)(n+1)a_{n+2} = 0$$

$$a_{n+2} = a_n \frac{\ell(\ell+1) - n(n+1)}{(n+1)(n+2)}$$

Generally $\frac{a_{n+2}}{a_n} \to -1$ if $n \to \infty$

note: this power series does not Converge for $x = \pm 1$ Unless ℓ is an integer. Then series terminates when $n = \ell$ Since $a_n = 0$ for all $n > \ell$

solution w/ integer ℓ is denoted $P_\ell(x)$ & is a polynomial of degree ℓ

b) Properties of Legendre Polynomials

1) if l is even all coeff a_n w/ n odd are zero
" " " odd " " " " " even " "

$$P(-x) = (-1)^l P(x)$$

2) To normalize for all orders $P_l(1) = 1$

3) Complete orthogonal set on $-1 \le x \le 1$

$$\int_{-1}^{1} dx\, P_l(x) P_{l'}(x) = \delta_{ll'} \frac{2}{2l+1}$$

4) $P_0(x) = 1$

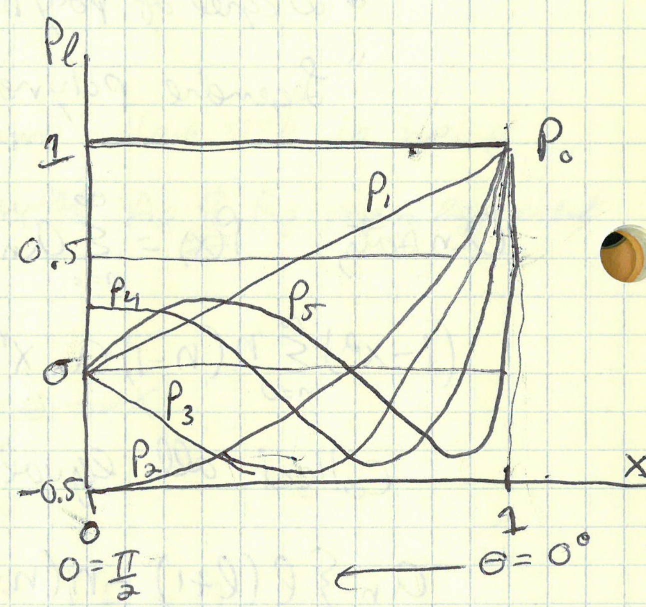
$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$



5) Rodrigues' formula

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l$$

6) Generating function

$$F(x, \mu) = \frac{1}{(1 - 2x\mu + \mu^2)^{1/2}} = \sum_{l=0}^{\infty} \mu^l P_l(x)$$

i.e. expanding as a power series in μ, coefficients are polynomials in μ

7) Recursion relations

$$(\ell+1) P_{\ell+1}(x) = (2\ell+1)\, x\, P_\ell(x) - \ell P_{\ell-1}(x)$$

$$(1-x^2)\frac{dP_\ell}{dx} = -\ell x P_\ell(x) + \ell P_{\ell-1}(x)$$

Summary: General solution of Laplace's eq w/ axial symmetry

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

Fix A_ℓ, B_ℓ using boundary conditions

note: if Φ is continuous at $r=0$ $B_\ell = 0$

for Φ corresponding to localized source $A_\ell = 0$

note: Similar to multipole expansion. Same $\measuredangle$ dependence for given order in expansion B_0 = monopole, B_1 = dipole, etc.

Spherical Harmonics $m \neq 0$

$$\frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right) + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)P = 0$$

Solutions are called "associated Legendre functions"

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell \qquad m = 0, \pm1, \pm2 \ldots \pm\ell$$

- orthogonality: $\int_{-1}^{1} dx\, P_\ell^m(x) P_{\ell'}^{m}(x) = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell\ell'}$

- Together with $e^{im\phi}$ the associated Legendre functions are complete on $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$

Spherical Harmonics:

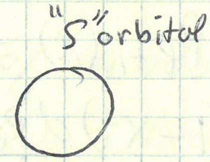
$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\phi}$$

$$\int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \; Y_\ell^m(\theta,\phi)\, Y_{\ell'}^{m'*}(\theta,\phi) = \delta_{\ell\ell'}\delta_{mm'}$$

$*$ = Complex Conjugate

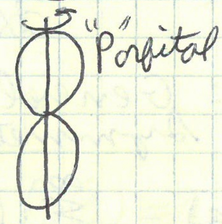
$$Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$$

$|Y_0^0|^2$



$$Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

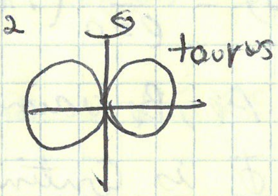
$|Y_1^0|^2$



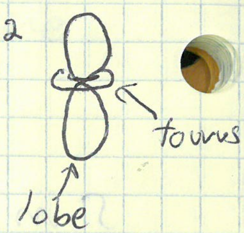
$$Y_1^{\pm 1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$$

etc.

$|Y_1^1|^2$



$|Y_2^0|^2$



$|Y_2^1|^2$

$$\Rightarrow \Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(A_\ell^m r^\ell + B_\ell^m \frac{1}{r^{\ell+1}}\right) Y_\ell^m(\theta,\phi)$$

Review & examples

0) Laplace's equation for spherical polar coords (r,θ,ϕ)

$$\Delta\Phi = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial\Phi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Phi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

a) Find special solution of the form

$$\Phi(r,\theta,\phi) = R(r)\, P(\cos\theta)\, Q(\phi)$$

$$Q_m(\phi) = e^{im\phi} \qquad m = 0, \pm1, \pm2 \ldots$$

$$R_\ell(r) = r^\ell \text{ or } \frac{1}{r^{\ell+1}} \qquad \ell = |m|, |m|+1, \ldots$$

P satisfies (w/ $x = \cos\theta$)

$$\frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right) + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)P = 0$$

if $m = 0$

$P = P_\ell(\cos\theta)$ ℓ^{th} Legendre Polynomial w/ $\ell = 0, 1, 2$

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$$

b) General solution for $m=0$ case (no ϕ dependence)

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

- P_ℓ are complete & orthogonal on $-1 \leq \cos\theta \leq 1$

$$\int_{-1}^{1} d\cos\theta \; P_\ell(\cos\theta) P_{\ell'}(\cos\theta) = \delta_{\ell\ell'} \frac{2}{2\ell+1}$$

- $P_0 \cos\theta = 1$

  $P_1 \cos\theta = \cos\theta$

  $P_2 \cos\theta = \frac{1}{2}(3\cos^2\theta - 1)$

note Similarity to multipole expansion. Same angular dependence for a given order in the expansion

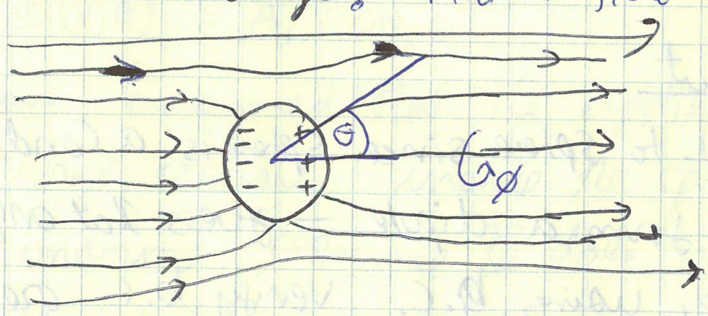
A_ℓ are zero for a localized source

c) if $m \neq 0$ we need spherical Harmonics

Examples:

1) Conducting sphere of radius a in uniform electric field. Sphere alters the field locally but far from sphere we expect no change. Find $\vec{E}$ field Everywhere

$\vec{E} = \vec{E}_0 \hat{e}_z$



a) Axial symmetry means $m=0$

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

b) for $r \to \infty$ $\vec{E} = E_0 \hat{z}$ $\quad \Phi = -E_0 z = -E_0 r\cos\theta$

Comparing w/ expansion for Φ

$A_\ell = 0$ if $\ell \neq 1$ $\quad A_1 = -E_0$

So $\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) - E_0 r \cos\theta$

c) Match B using boundary conditions at sphere.

Φ is constant at $r=a$ lets make it $\Phi=0$ (for E field constants don't matter)

Use orthogonality of the P_ℓ to find B_ℓ

$$\Phi(a,\theta) = -E_0 a P_1 + \sum_{\ell=0}^{\infty} \frac{B_\ell}{a^{\ell+1}} P_\ell = 0$$

integrate both sides with $\int_{-1}^{1} d\cos\theta \, P_{\ell'}$

$$E_0 a \int_{-1}^{1} P_1(x) P_{\ell'}(x)\,dx = \sum_{\ell=0}^{\infty} \frac{B_\ell}{a^{\ell+1}} \int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\,dx$$

$$E_0 a \frac{2}{2\ell'+1} \delta_{1\ell'} = \frac{B_{\ell'}}{a^{\ell'+1}} \frac{2}{2\ell'+1}$$

$$B_\ell = E_0 a^{(\ell+2)} \delta_{1\ell} \qquad B_1 = E_0 a^3 \qquad B_{\ell\neq 1} = 0$$

$$\Phi(r,\theta) = -E_0 \left(1 - \frac{a^3}{r^3}\right) r\cos\theta$$

d) <u>Short Cut</u>

Field is $\perp$ to sphere since sphere is a conductor

Charges form a dipole — guess that only $\ell=1$ contributes

match B_1 using B.C. Verify b.C. are met for all θ

& invoke uniqueness to ensure you have <u>the</u> solution

e) Electric field $\qquad$ inside sphere $\vec{E}=0$

outside
$$\begin{cases} E_r = -\frac{\partial \Phi}{\partial r} = E_0 \left(1 + \frac{2a^3}{r^3}\right)\cos\theta \\ E_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = -E_0\left(1 - \frac{a^3}{r^3}\right)\sin\theta \end{cases}$$

note at $r=a$ $\quad E_\theta = 0$

Surface Charge density is $E_r(r=a) = 4\pi \rho_s$

used $(\epsilon_2 E_2 - \epsilon_1 E_1) \cdot n = 4\pi \rho_s$ $\quad \rho_s = \frac{3}{4\pi} E_0 \cos\theta$

& the fact that E_1 inside sphere is zero & $\epsilon_2 = 1$

$$\Phi = \underbrace{-E_0 r \cos\theta}_{\text{External Contribution}} + \underbrace{\frac{E_0 a^3 \cos\theta}{r^2}}_{\text{local induced Contribution}} \qquad r>a$$

Φ induced is just the dipole potential with $\vec{P} = \vec{E}_0 a^3$

Ex 2 lets repeat this for a linear dielectric sphere

a) axial symmetry $\Rightarrow \Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}}\right) P_\ell(\cos\theta)$

b) for $r \to \infty$ $\vec{E} = E_0 e_z \Rightarrow \Phi = -E_0 z = -E_0 r \cos\theta$

c) Guess: need $\ell = 1$ only

$$r>a: \Phi(r,\theta) = -E_0 r\cos\theta + \underbrace{\frac{B_1}{r^2}}_{\text{This time dipole will not Completely Cancell field inside}} \cos\theta$$

$$r<a: \Phi(r,\theta) = A_1' r \cos\theta$$

Here we have assumed that the applied field does not lead to a charge density inside the sphere (i.e. polarization of sphere is uniform). Otherwise Φ does not obey the Laplace eq for $r<a$

b.c. at $r=a$

- $D_\perp = D_r$ Continuous
- $E_\parallel = E_\theta$ Continuous
- Φ Continuous

Φ at a

$$-E_0 a \cos\theta + \frac{B_1}{a^2}\cos\theta = A_1' a \cos\theta$$

Φ continuous ensures $E_\parallel$ Continuous