Surface charge density is $E_r(r=a) = 4\pi \rho_s$

used $(\epsilon_2 E_2 - \epsilon_1 E_1)\cdot n = 4\pi\rho_s$ $\qquad \rho_s = \frac{3}{4\pi} E_0 \cos\theta$

& the fact that E_1 inside sphere is zero & $\epsilon_2 = 1$

$$\Phi = \underbrace{-E_0 r\cos\theta}_{\text{External Contribution}} + \underbrace{\frac{E_0 a^3 \cos\theta}{r^2}}_{\text{local induced Contribution}} \qquad r>a$$

Φ induced is just the dipole potential with $\vec{P} = \vec{E}_0 a^3$

Ex 2 lets repeat this for a linear dielectric sphere

a) axial symmetry $\Rightarrow \Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}}\right) P_\ell(\cos\theta)$

b) for $r\to\infty$ $\vec{E} = E_0 e_z \Rightarrow \Phi = -E_0 z = -E_0 r\cos\theta$

c) Guess: need $\ell = 1$ only

$$r>a: \Phi(r,\theta) = -E_0 r\cos\theta + \underbrace{\frac{B_1}{r^2}}_{\text{This time dipole will not Completely cancell field inside}}\cos\theta$$

$$r<a: \Phi(r,\theta) = A_1' r\cos\theta$$

Here we have assumed that the applied field does not lead to a charge density inside the sphere (i.e. polarization of sphere is uniform). Otherwise Φ does not obey the Laplace eq for $r<a$

b.c. at $r=a$

- $D_\perp = D_r$ Continuous
- $E_\parallel = E_\theta$ Continuous
- Φ Continuous

Φ at a

$$-E_0 a\cos\theta + \frac{B_1}{a^2}\cos\theta = A_1' a\cos\theta$$

Φ continuous ensures $E_\parallel$ continuous

$$D_\perp = \epsilon E_r = -\epsilon \frac{d\Phi}{dr} \quad \text{if } r<a$$

$$D_\perp = E_r = -\frac{d\Phi}{dr} \quad \text{if } r>a$$

$$E_0 \cos\theta + \frac{2B_1}{a^3}\cos\theta = -\epsilon A_1' \cos\theta$$

Solve: $A_1' = \dfrac{-3E_0}{2+\epsilon}$ $\quad B_1 = \dfrac{\epsilon-1}{\epsilon+2} a^3 E_0$

$$\Phi = \begin{cases} -E_0 r\cos\theta + \dfrac{\epsilon-1}{\epsilon+2}\dfrac{a^3}{r^2} E_0 \cos\theta & r>a \\ -\dfrac{3E_0 r\cos\theta}{2+\epsilon} & r<a \end{cases}$$

notes:
- if $\epsilon = 1$ $\vec{E} = \vec{E}_0$ or $\Phi = -E_0 e_z$ for all space
- for $r>a$ $\Phi = \Phi_{external} + \Phi_{locally\ induced}$

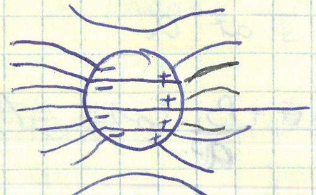
  $\Phi_{induced}$ is Potential of dipole $\vec{p} = \dfrac{\epsilon-1}{\epsilon+2} a^3 \vec{E}_0$

- if $\epsilon \to \infty$ recover dipole of Conducting sphere
- for $r<a$ $\Phi = -\dfrac{3E_0 z}{2+\epsilon} \Rightarrow \vec{E} = \dfrac{3\vec{E}_0}{2+\epsilon}$

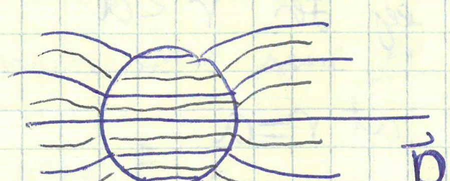
for linear dielectric $\quad \vec{D} = \epsilon\vec{E} = \dfrac{3\epsilon E_0}{2+\epsilon}$ if $r<a$ $\quad \vec{P} = \dfrac{1}{4\pi}(\vec{D}-\vec{E}) = \dfrac{3}{4\pi}\dfrac{\epsilon-1}{\epsilon+2}\vec{E}_0$

$$= \frac{\vec{p}}{\frac{4}{3}\pi a^3}$$

dielectric cannot cancel all the E field lines



Polarization charges are sources & sinks for E



all field lines are continuous & do not terminate

Remark: The fact that we found a solution confirms our assumption that there is no bound charge density inside sphere. This is not general for arbitrary shapes. It only works for ellipsoids

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Laplace Eq in Cylindrical Coordinates r, θ, z

$$\Delta \Phi = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

a) look for special solutions $\Phi(r,\theta,z) = R(r)\,Q(\theta)\,Z(z)$ just like before:

$Q(\theta) = e^{in\theta} \qquad n = 0, \pm 1, \pm 2 \ldots$

$Z(z) = e^{kz}$

remaining eq for $R(r)$

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(k^2 r^2 - n^2\right) R = 0$$

b) First $K = 0$ case. Translational symmetry in z direction

$$R_n(r) = \begin{cases} A_0 + B_0 \ln r & \text{if } n = 0 \\ A_n r^n + \dfrac{B_n}{r^n} & \text{if } n = \pm 1, \pm 2, \ldots \end{cases}$$

General Solution:

$$\Phi(r,\theta) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty}\left[A_n r^n + \frac{B_n}{r^n}\right]\cos(n\theta)$$

$$+ \sum_{n=1}^{\infty}\left[C_n r^n + D_n \frac{1}{r^n}\right]\sin(n\theta)$$

note: This is different from eq. 3.79 in Heald & Marion. Their expression does not span the space of all solutions aside:

$A_n C_n = a$ ratios give you only three parameters

$B_n D_n = b$ $\frac{a}{b} = \frac{c}{d}$ so there is a dependence.

$B_n C_n = c$

$A_n D_n = d$

c) General Case $k \neq 0$

define $u = |k| r$ then $\frac{d}{dr} = |k| \frac{d}{du}$

$$\frac{1}{u}\frac{d}{du}\left(u\frac{dR}{du}\right) + \left(1 - \frac{n^2}{u^2}\right)R = 0$$

or

$$u^2\frac{d^2R}{du^2} + u\frac{dR}{du} + (u^2 - n^2)R = 0 \quad \text{Bessel's eq.}$$

Solution: Ansatz $R(u) = u^b \sum_{m=0}^{\infty} a_m u^m$ w/ $a_0 \neq 0$

Plug into diff eq collect powers of u & get recursion relation for the a_m

$$((m+b)^2 - n^2)a_m = -a_{m-2}$$

analyse recursion relation

- $m=0$: $(b^2 - n^2)a_0 = a_{-2}$ but $a_0 \neq 0$ while $a_{-2} = 0$

  must have $b = \pm n$ we choose $b = n$ & allow for positive & negative n

- $m=1$: $((b+1)^2 - n^2)a_1 = a_{-1}$ but $a_{-1} = 0$ $(b+1)^2 - n^2 \neq 0$

  if $b = \pm n \Rightarrow a_1 = 0$ Similarly all a_m w/ m odd are 0

- Even m $a_m = \frac{-a_{m-2}}{(m+b)^2 - n^2} = \frac{-a_{m-2}}{m(2n+m)}$ with $b = n$

Convention: Take $a_0 = \frac{1}{2^n n!}$