Surface charge density is \( \sigma \) for \( r = a \):

\[ \sigma = 4\pi \frac{1}{2} \]

and

\[ \n = 4\pi \frac{3}{4} E_0 \cos \theta \]

So the total charge on the sphere is zero if \( n = 1 \).

\[ \Phi = -E_0 \cos \theta + \frac{E_0 a^3 \cos \theta}{r^2} \quad r > a \]

**External Contribution**

\[ \Phi \text{ induced is just the dipole potential with } \vec{p} = E_0 a^3 \]

**Local induced Contribution**

Ex 2: Let's repeat this for a dielectric sphere.

a) Axial symmetry \( \Rightarrow \Phi(r, \theta) = \frac{2}{e_0} \left( A e^{-l} + B e^{l} \right) \rho(r, \theta) \)

b) For \( r > a \): \( \vec{E} = E_0 \vec{e}_z \Rightarrow \Phi = -E_0 \vec{e}_z = -E_0 \cos \theta \)

c) Guess: need \( l = 1 \) only

\[ r > a : \Phi(r, \theta) = -E_0 \cos \theta + \frac{B_1}{r^2} \cos \theta \]

This time the dipole will not completely cancel the field inside.

\[ r < a : \Phi(r, \theta) = A' r \cos \theta \]

Here we have assumed that the applied field does not lead to a charge density inside the sphere (i.e. polarization of sphere is uniform). Otherwise \( \Phi \) does not obey the Laplace eq for \( r < a \).

**b.c. at \( r = a \)**

- \( D_1 = D_T \) Continuous
- \( E_z = E_0 \) Continuous
- \( \Phi \) Continuous

**\( \Phi \) at \( r = a \)**

\[ -E_0 a^3 \cos \theta + \frac{B_1}{a^3} \cos \theta = A' a \cos \theta \]

\[ \Phi \text{ continuous and } E_z \text{ continuous} \]
\[ D_r = \varepsilon E_r = -\frac{\varepsilon d\Phi}{dr} \quad \text{if } r < a \]
\[ D_r = \varepsilon E_r = -\frac{d\Phi}{dr} \quad \text{if } r > a \]

\[ E_0 \cos \theta + 2B_1 \cos 2\theta = -\varepsilon A'_1 \cos \theta \]

Solve: \[ A'_1 = -\frac{3E_0}{2+\varepsilon} \quad B_1 = \frac{\varepsilon - 1}{2+\varepsilon} a^2 E_0 \]

\[ \Phi = \begin{cases} -\frac{3E_0 r \cos \theta}{2+\varepsilon} & \text{if } r > a \\ -\frac{3E_0 r \cos \theta}{2+\varepsilon} & \text{if } r < a \end{cases} \]

Notes:  
- If \( \varepsilon = 1 \), \( \vec{E} = \vec{E}_0 \) or \( \Phi = -E_0 r \) for all space
- For \( r > a \), \( \Phi = \Phi_{\text{external}} + \Phi_{\text{locally induced}} \)

\( \Phi_{\text{induced}} \) is potential of dipole \( \vec{P} = \frac{\varepsilon - 1}{2+\varepsilon} a^3 E_0 \)

- If \( \varepsilon \to 0 \), recover dipole of conducting sphere
- For \( r < a \), \( \Phi = \frac{-3E_0 z}{2+\varepsilon} \Rightarrow \vec{E} = \frac{3E_0}{2+\varepsilon} \)

On linear dielectric \( \vec{D} = \varepsilon \vec{E} = \frac{3\varepsilon E_0}{2+\varepsilon} \) if \( r < a \)

\( \vec{P} = \frac{1}{4\pi}(\vec{D} - \vec{E}) = \frac{3}{4\pi} \varepsilon - 1 \vec{E}_0 \)

\( \frac{\vec{P}}{\frac{4}{3} \pi a^3} \)

Dielectric cannot contain all the \( E \) field lines

Polarization Charges at Sources & Sinks for \( \vec{E} \)

All field lines are continuous & do not terminate
Renew: the fact that we found a solution confirms our assumption that there is no bound charge density inside the sphere. This is not general for arbitrary shapes. It only works for ellipsoids.

Solve Eq. in cylindrical coordinates \( r, \theta, z \)

\[
\Delta \Phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0
\]

a) Look for special solutions \( \Phi(r, \theta, z) = R(r) \Theta(\theta) \Phi(z) \)

just like before:

\( \Theta(\theta) = e^{in\theta} \quad n = 0, \pm 1, \pm 2, \ldots \)

\( \Phi(z) = e^{ikz} \)

remaining eq for \( R(r) \)

\[
r \frac{d}{dr} \left( r \frac{dR}{dr} \right) + \left( k^2 r^2 - n^2 \right) R = 0
\]

b) First \( K = 0 \) case. Translational symmetry in \( z \) direction

\[
R_n(r) = \sum A_0 + B_0 \ln r \quad \text{if } n = 0
\]

\[
A_n r^n + \frac{B_n}{r^n} \quad \text{if } n = \pm 1, \pm 2, \ldots
\]

General Solution:

\[
\Phi(r, \theta, z) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} \left[ A_n r^n + \frac{B_n}{r^n} \right] \cos(n\theta)
\]

\[
+ \sum_{n=1}^{\infty} \left[ C_n r^n + D_n \frac{1}{r^n} \right] \sin(n\theta)
\]

Note: This is different from Eq. 3.79 in HEALTH & MARIN. Their expression does not span the space of all solutions.
Aside:

\[ A_n C_n = a \]
\[ B_n D_n = b \]
\[ B_n C_n = c \]
\[ A_n D_n = d \]

The ratios give you only three parameters
\[ \frac{a}{b} = \frac{c}{d} \] so there is a dependence.

c) General Case \( b \neq 0 \)

Define \( U = |k| r \) Then
\[ \frac{d}{dr} \left( \frac{dR}{du} \right) + \left( -\frac{n^2}{u^2} \right) R = 0 \]

\[ \frac{1}{U} \frac{d}{du} (U dR) + \left( -\frac{n^2}{u^2} \right) R = 0 \]

Bessel's Eq.

Solution: Ansatz
\[ R(u) = u^b \sum_{m=0}^{\infty} a_m u^m \] w/ \( a_0 \neq 0 \)

Plug into diff eq collect powers of \( u \) & get recursion relation for the \( a_m \)

\[ ((m+b)^2 - n^2) a_m = -a_{m-2} \]

Analyze recursion relation

\[ m = 0 : \quad (b^2 - n^2) a_0 = a_{-2} \]
but \( a_0 \neq 0 \) while \( a_{-2} = 0 \)

Must have \( b = \pm n \) we choose \( b = n \) & allow for positive & negative \( n \)

\[ m = 1 : \quad (b + 1)^2 - n^2) a_1 = a_{-1} \]
but \( a_{-1} = 0 \) \( (b+1)^2 - n^2 \neq 0 \)

If \( b = \pm n \) \( \Rightarrow a_1 = 0 \)
Similarly all \( a_m \) w/ \( m \) odd are 0

Even \( m \)

\[ a_m = \frac{-a_{m-2}}{(m+b)^2 - n^2} = \frac{-a_{m-2}}{m(2m+n)} \]

Convention: Take \( A_0 = \frac{1}{n!} \)