$$R_n(u) = J_n(u) = \left(\frac{u}{2}\right)^n \sum_{\lambda=0}^{\infty} \frac{(-1)^\lambda}{\lambda!(\lambda+n)!}\left(\frac{u}{2}\right)^{2\lambda}$$ "Bessel function"

note: for non integer n factorial is defined via Γ function

$$\Gamma(n+1) = n! = \int_0^\infty dx\, x^n e^{-x}$$ This applies to positive & negative n

For non integer n: J_n & J_{-n} are linearly independent Solutions. However, if n is integer

$$J_n(u) = (-1)^n J_{-n}(u)$$

The Bessel functions give only one indep solution.
The second indep Solution is

$$N_n(u) = \lim_{n' \to n} \frac{J_{n'}(u) \cos n'\pi - J_{-n'}(u)}{\sin n\pi}$$

Neumann function of order n"
Sometimes called Bessel function of 2nd kind Sometimes denoted $Y_n(u)$

General Solution to Laplace's eq in Cylindrical Coords

$$\Phi(r,\theta,z) = \sum_{n} \sum_{k\pm} \left(A^{\pm}_{nk} J_n(kr) + B^{\pm}_{nk} N_n(kr)\right) e^{in\theta} e^{\pm kz}$$

$n = 0, \pm1, \pm2 \ldots$

$k > 0$

$$+ \sum_{n=1}^{\infty} \left\{ \left(A_n r^n + B_n \frac{1}{r^n}\right) \cos n\theta + \left(C_n r^n + \frac{D_n}{r^n}\right) \sin n\theta \right.$$

$$+ A_0 + B_0 \ln r$$

Review & Examples

0) Laplace Eq in Cylindrical Coords

$$\Delta \Phi = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Special Solutions of the form $\Phi(r,\theta,z) = R(r)\, Q(\theta)\, Z(z)$

$Q(\theta) = e^{in\theta}$ n integer

$Z(z) = e^{\pm kz}$

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (k^2r^2 - n^2)R = 0$$

if $k=0$ $R_n(r) = \begin{cases} A_0 + B_0 \ln r & \text{if } n=0 \\ A_n r^n + B_n \frac{1}{r^n} & \text{else} \end{cases}$

if $k \neq 0$ $R_n(r) = J_n(kr)$ Bessel function

or $N_n(kr)$ Neumann function

$J_n(u)$ = is solution to the equation:

$$u^2\frac{d^2 J_n}{du^2} + u\frac{dJ_n}{du} + (u^2 - n^2) J_n = 0$$

$$J_n(u) = \left(\frac{u}{2}\right)^n \sum_{\lambda=0}^{\infty} \frac{(-1)^\lambda}{\lambda!(\lambda+n)!}\left(\frac{u}{2}\right)^{2\lambda}$$

for non-integer $n^{\pm}$, J_n & J_{-n} are two linearly indep solutions

for integer n (which is physically relevant case) the second linearly indep solution is

$$N_n(u) = \lim_{n' \to n} \frac{J_{n'}(u)\cos n'\pi - J_{-n'}(u)}{\sin n'\pi}$$

Neumann function or Bessel function of 2nd kind

~~General Solution~~:

General Solution:

$$\Phi(r,\theta,z) = \sum_{n=1}^{\infty} \left\{ \left(A_n r^n + B_n \frac{1}{r^n}\right)\cos n\theta + \left(C_n r^n + D_n \frac{1}{r^n}\right)\sin n\theta \right\}$$

$$+ A_0 + B_0 \ln r + \sum_{n=-\infty}^{\infty} \sum_{k>0} \sum_{\pm} e^{in\theta} e^{\pm kz}\left(A^{\pm}_{nk} J_n(kr) + B^{\pm}_{nk} N_n(kr)\right)$$

Orthogonality & Completeness

With Sin & Cos we know $\sin kx$ for all k with $\sin(ka)=0$ are complete & orthogonal on $0 \le x \le a$. Note that this is not true for all $\sin kx$ w/ arbitrary k

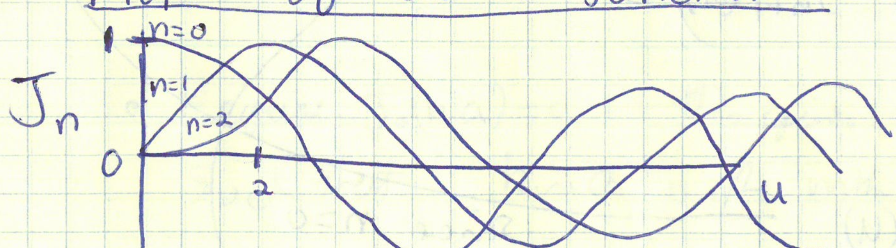
Here, the $J_n(kr)$ are complete & orthogonal on $0<r<a$ for those k where $J_n(ka)=0$. The integration measure for orthogonality is $r\,dr$ so

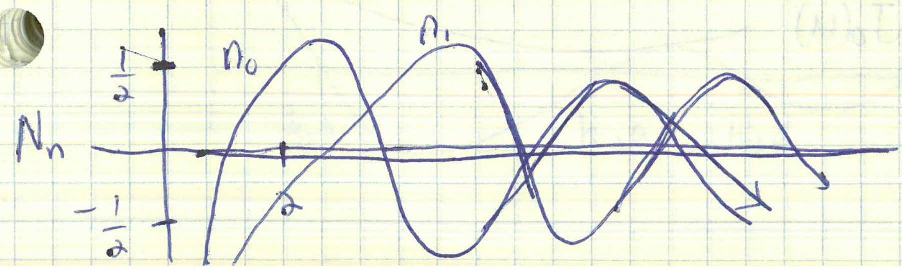
$$\int_0^a r\,dr\, J_n(kr) J_n(k'r) = \frac{a^2}{2} J_{n+1}^2(ka)\,\delta_{kk'}$$

with k, k' such that $J_n(ka) = J_n(k'a) = 0$

note: Orthogonality relation involves Bessel functions of the same n only. For different n, the factor $e^{in\theta}$ in the expression for Φ will ensure orthogonality of the special solutions

Properties of Bessel functions





N_0 & N_1 are interchanged in Heald & Marion

b) $J_n(u) = \frac{1}{n!}\left(\frac{u}{2}\right)^n + \dots$ if $u \ll 1$, $n \geq 0$

$$N_n(u) = \begin{cases} \frac{2}{\pi}\left[\ln\frac{u}{2} + 0.5772\dots\right] & n=0 \ \& \ u \ll 1 \\ -\frac{(n-1)!}{\pi}\left(\frac{2}{u}\right)^n & n>0 \ \& \ u \ll 1 \end{cases}$$

c) $J_n(u) \sim \sqrt{\frac{2}{\pi u}} \cos\left(u - \frac{n\pi}{2} - \frac{\pi}{4}\right)$ if $u \gg \frac{n\pi}{2}$, $n \geq 0$

$N_n(u) \sim \sqrt{\frac{2}{\pi u}} \sin\left(u - \frac{n\pi}{2} - \frac{\pi}{4}\right)$ if $u \gg \frac{n\pi}{2}$, $n \geq 0$

d) $J_{-n}(u) = (-1)^n J_n(u)$ for integer n

$N_{-n}(u) = (-1)^n N_n(u)$

e) $J_n(u) = \frac{u}{2n}\left(J_{n-1} + J_{n+1}\right)$

$N_n(u) = \frac{u}{2n}\left(N_{n-1} + N_{n+1}\right)$

$$\left.\begin{aligned} \frac{d}{du} J_n(u) &= \frac{n}{u} J_n(u) - J_{n+1}(u) \\ &= -\frac{n}{u} J_n(u) + J_{n-1}(u) \\ &= \frac{1}{2}\left(J_{n-1}(u) - J_{n+1}(u)\right) \end{aligned}\right\} \text{Same for } N_n$$

examples:

$\frac{d}{du} J_0(u) = -J_1(u)$ since $n=0$

$\frac{d}{du}\left[u J_1(u)\right] = u J_0(u)$

$$\frac{1}{k^2-k'^2}\frac{d}{du}\left\{k'u\,J_{n-1}(k'u)\,J_n(ku)-ku\,J_{n-1}(ku)\,J_n(k'u)\right\} =$$

$$u\,J_n(k'u)\,J_n(ku)$$

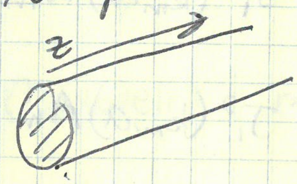
if $k = k'$ you get a singularity. If $k \neq k'$ you get zero. since the function is evaluated at 0 & ka where the system has nodes.

relation proves orthogonality

$$\frac{d}{du}\left\{\frac{u^2}{2}\left(J_n(u)^2 + J_{n+1}(u)^2\right) - n\,u\,J_n(u)\,J_{n+1}(u)\right\} = u\,J_n^2(u)$$

This relation proves normalization

3) Example:



Conducting Cylinder

wall at $\Phi = 0$ end at $z=0$ $\epsilon_0/\Phi = \Phi_0$

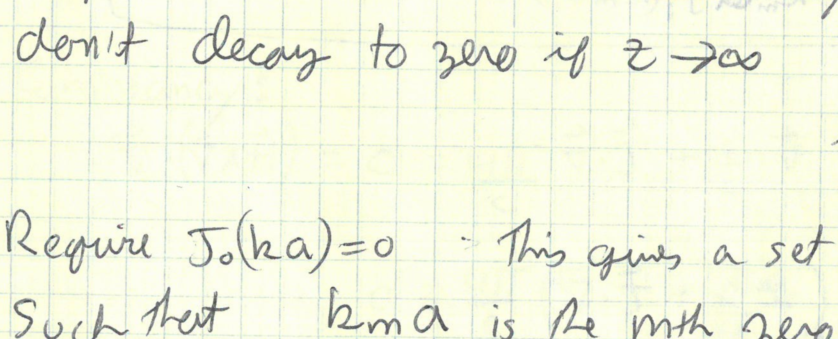
find Φ inside Cylinder

- No dependence on θ $n=0$ only

$$\Phi = \underbrace{A_0 + B_0 \ln r}_{n=0,\ K=0 \text{ part}} + \underbrace{\sum_{k,\pm} A^{\pm}_{k0}\, J_0(kr)\, e^{-kz}}_{n=0\quad k>0 \text{ part.}}$$

- keep "+" terms with $k>0$ only all other terms don't decay to zero if $z \to \infty$ $A_0 = 0$, $B_0 = 0$ since $\ln r$ diverges at $r=0$

- Require $J_0(ka) = 0$ · This gives a set of discrete k_m such that $k_m a$ is the mth zero of J_0 $m = 1, 2, \ldots$



$$\Phi(r,\theta,z) = \sum_m A_m J_0(k_m r)\, e^{-k_m z}$$

Apply boundary condition at $z=0$

$$\Phi(r,\theta,0) = \Phi_0 \quad \text{is} \quad r \le a$$

$$\Phi_0 = \sum_m A_m J_0(k_m r)$$

$$\int_0^a dr\, r\, J_0(k_{m'} r)\, \Phi(r,\theta,0) = \sum_m A_m \int_0^a dr\, r\, J_0(k_{m'} r) J_0(k_m r)$$

$$\Phi_0 \int_0^a dr\, r\, J_0(k_{m'} r) = \frac{a^2}{2} J_1^2(k_{m'} a) A_{m'}$$

$$\Phi_0 \frac{1}{k_{m'}^2} \int_0^{k_{m'} a} du\, u J_0(u) = \frac{a^2}{2} J_1^2(k_{m'} a) A_{m'}$$

$$\Phi_0 \frac{1}{k_{m'}^2} k_{m'} a J_1(k_{m'} a) = \frac{a^2}{2} J_1^2(k_{m'} a) A_{m'}$$

$$A_{m'} = \frac{2\Phi_0}{k_{m'} a J_1(k_{m'} a)}$$

$$\boxed{\Phi(r,\theta,z) = \sum_m \frac{2\Phi_0}{k_m a J_1(k_m a)} J_0(k_m r)\, e^{-k_m z}}$$

Maxwells' Equations for time dependent fields

a) Faraday's Law

$$\underbrace{\oint_{\Gamma} \vec{E}\cdot d\vec{\ell}}_{\substack{\text{EMF}\\ \text{electromotive force}}} = \underbrace{-\frac{1}{c}\int_S \frac{\partial \vec{B}}{\partial t}\cdot \hat{n}\, da}_{\substack{\text{Time derivative of}\\ \text{enclosed magnetic flux}}}$$

Demo of wire w/bulb through Magnet to illustrate Faraday's Law.

introduced Extra Credit reading

or

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

b) Maxwell's modification of Ampere's law

$$\int_{\Gamma} \vec{H}\cdot d\vec{\ell} = \underbrace{\frac{4\pi}{c}\int_S \vec{J}\cdot d\vec{a}}_{\text{linked Current}} + \underbrace{\frac{1}{c}\frac{d}{dt}\int_S \vec{D}\cdot d\vec{a}}_{\text{displacement}}$$

$\boxed{\text{Maxwell added this part}}$

or

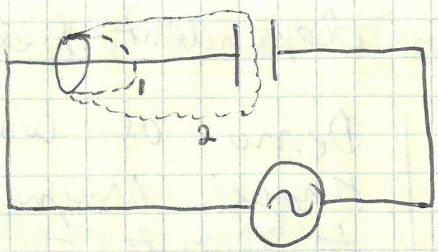
$$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$

for Consistancy:

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = 0 = \frac{4\pi}{c}\vec{\nabla}\cdot\vec{J} + \frac{1}{c}\vec{\nabla}\cdot\frac{\partial \vec{D}}{\partial t} \qquad \vec{\nabla}\cdot\vec{D} = 4\pi\rho_f$$

$$0 = \frac{4\pi}{c}\underbrace{\left(\vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}\rho_f\right)}_{\text{Conservation of charge}}$$

The Book goes through a nice argument or example that illustrates why this ~~term~~ Displacement term needs to be added



Case 1 $J > 0$

Case 2 D makes up for J being zero

Case 1 : Surface goes through wire

$$\oint_T \vec{H}\cdot d\vec{\ell} = \frac{4\pi}{c} I = \frac{4\pi}{c}\frac{dq}{dt}$$

Case 2 Surface goes through empty space between capacitor plates

$$\oint_T \vec{H}\cdot d\vec{\ell} = \underbrace{\frac{1}{c}\int_S \frac{d\vec{E}}{dt}\, da}_{\text{since } \vec{D}=\vec{E} \text{ in } \text{~~vacuum~~ free space}} = \frac{4\pi}{c}\frac{dq}{dt}$$

by Gauss' law $\int \vec{E}\cdot d\vec{a} = 4\pi q$

c) The remaining fundamental eqs.

$$\left.\begin{aligned} \vec{\nabla}\cdot\vec{B} &= 0 \\ \vec{\nabla}\cdot\vec{D} &= 4\pi\rho_f \end{aligned}\right\}$$ remain valid in the time dependent case

For Consistency : Take divergence of $-\frac{1}{c}\frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times\vec{E}$

$$-\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B}) = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{E}) = 0 \checkmark$$

d) Maxwell's eq for time dependent fields

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho_f \quad \text{or} \quad \vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J}_f \quad \text{or} \quad \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$$

Potential functions

a) $\vec{\nabla} \cdot \vec{B} = 0$ for every time t there exists $\vec{A}(\vec{r}, t)$ such that

$$\vec{B} = \vec{\nabla} \times \vec{A}(\vec{r}, t)$$

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = \nabla \times \left(E + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \right) = 0$$

Therefore, for every t there exists $\Phi(\vec{r}, t)$ such that

$$\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\nabla \Phi \quad \text{or} \quad \vec{E} = -\nabla \Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

b) Φ & $\vec{A}$ are not unique

Recall that for time <u>independent</u> case

$$\left.\begin{aligned} &\bullet\ \Phi \to \Phi + \text{Const} \\ &\vec{A} \to \vec{A} + \vec{\nabla} \xi \end{aligned}\right\} \text{left } E \,\&\, B \text{ unchanged}$$

For the time <u>dependent</u> case

$\vec{A} \to \vec{A} + \nabla \xi(\vec{r}, t)$ gives the same B

But

$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla \Phi$ is changed

Solution is to simultaneously change

$\Phi \to \Phi - \frac{1}{c}\frac{\partial \xi}{\partial t}$ Then both E & B are left unchanged

$$\boxed{\begin{aligned} &\vec{A} \to \vec{A} + \nabla \xi \\ &\Phi \to \Phi - \frac{1}{c}\frac{\partial \xi}{\partial t} \end{aligned}}$$

This is called a "gauge transformation"
all physical observables are invarient under gauge transformations
"Gauge invarient"

(5) It may be convenient to "fix a gauge" to impose additional conditions on $\vec{A}, \Phi$ while keeping the ability to describe all $\vec{E}, \vec{B}$. This often helps w/ calculations

Well known gauges:

"Lorentz Gauge" $\vec{\nabla}\cdot\vec{A} = -\frac{1}{c}\frac{\partial\Phi}{\partial t}$

"Coulomb Gauge" $\vec{\nabla}\cdot\vec{A} = 0$

"London Gauge"

The Lorentz Gauge still leaves some freedom for $\vec{A}$ & Φ:

$$\vec{A} \to \vec{\nabla}\xi + \vec{A} \qquad \Phi \to \Phi - \frac{1}{c}\frac{\partial\xi}{\partial t}$$

if $\vec{\nabla}^2\xi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\xi = 0$ this transformation is still compatible w/ the Lorentz gauge

Need an additional boundary condition to fully fix $\vec{A}, \Phi$

The Lorentz gauge turns everything into wave equations

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$
$$= \nabla\cdot\left(-\nabla\Phi - \frac{1}{c}\frac{\partial A}{\partial t}\right)$$
$$= -\Delta\Phi - \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A})$$

Lorentz gauge $\vec{\nabla}\cdot\vec{A} = -\frac{1}{c}\frac{\partial\Phi}{\partial t}$

$$= -\Delta\Phi + \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 4\pi\rho$$

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial E}{\partial t} = 4\pi J$$
$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) + \frac{1}{c}\frac{\partial}{\partial t}\nabla\Phi + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}$$

Lorentz gauge $\vec{\nabla}\cdot\vec{A} = -\frac{1}{c}\frac{\partial\Phi}{\partial t}$

$$= \underbrace{-\Delta\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A})}_{\text{BAC CAB Rule}} - \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) + \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}$$
$$= \Delta\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{J}$$

$$\left.\begin{aligned}\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi &= -4\pi\rho \\ \left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} &= -\frac{4\pi}{c}\vec{J}\end{aligned}\right\}$$ for static fields Lorentz gauge becomes Coulomb gauge

$$\Phi = \int_V dv' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \qquad \vec{A} = \int_V dv' \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

– Energy density & the Pointing Vector

1) Energy

a) Recall how energy was introduced in Classical Mech.

$$m\frac{d}{dt}\frac{d\vec{r}}{dt} = -\frac{d}{dr}V \quad \text{(conservative force only)}$$

or

$$m\frac{d}{dt}\frac{d\vec{r}}{dt} = \vec{F}_{\text{non cons}} - \frac{d}{dr}V \quad \text{where } V \text{ is potential energy}$$

multiply by $\frac{d\vec{r}}{dt}$:

$$m\left(\frac{d}{dt}\frac{d\vec{r}}{dt}\right)\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\tfrac{1}{2}mv^2\right) \qquad \vec{v}=\frac{d\vec{r}}{dt}$$

$$= -\frac{dV}{dr}\cdot\frac{d\vec{r}}{dt} + \vec{F}_{nc}\cdot\frac{d\vec{r}}{dt}$$

$$= -\frac{d}{dt}V + \vec{F}_{nc}\cdot\vec{v}$$

$$\Rightarrow \frac{d}{dt}\Big(\underbrace{\tfrac{1}{2}mv^2}_{\text{K.E.}} + \underbrace{V}_{\text{P.E.}}\Big) = \underbrace{\vec{F}_{nc}\cdot\vec{v}}_{\text{work done on particle/unit time}}$$

b) Lets repeat this analysis for electric & Magnetic fields. Instead of Energy we will discuss Energy density $\mathcal{E}$

Conservation Laws

$$\frac{\partial}{\partial t}[\text{energy density}] + \vec{\nabla}[\text{energy current}] + \left[\begin{matrix}\text{work done by E\&M}\\ \text{field per unit vol}\end{matrix}\right] = 0$$

Can we rewrite maxwell's equations in this form?

c) $\frac{1}{c}\frac{\partial\vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E}$ multiply ~~by~~ w/ $\vec{H}$

$\frac{1}{c}\frac{\partial\vec{D}}{\partial t} = \vec{\nabla}\times\vec{H} - \frac{4\pi}{c}\vec{J}_f$ multiply w/ $\vec{E}$

add: $\frac{1}{c}\left(\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} + \vec{E}\frac{\partial\vec{D}}{\partial t}\right) = -\frac{4\pi}{c}\vec{J}_f\cdot\vec{E} + (-\vec{\nabla}\times\vec{E})\cdot\vec{H} + \vec{E}\cdot(\vec{\nabla}\times\vec{H})$

- $\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{H})$
- define $\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{H}$ "Poynting vector"
- Linear dielectric $\vec{B} = \mu\vec{H}$, $\vec{D} = \epsilon\vec{E}$

$$\boxed{\frac{\partial}{\partial t}\left[\frac{1}{8\pi}\left(\epsilon E^2 + \frac{1}{\mu}B^2\right)\right] + \vec{\nabla}\cdot\vec{S} + \vec{E}\cdot\vec{J}_f = 0 \quad \text{Pointing's thm}}$$

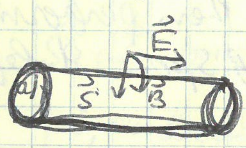
$\mathcal{E} = \frac{1}{8\pi}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})$ energy density

Energy current density or "power transmitted per cross sectional area"

Work done per unit time per volume (note: $\vec{B}$ does no work)

Integral form: $\frac{d}{dt}\int_V dv \frac{1}{8\pi}\left(\epsilon E^2 + \frac{1}{\mu}B^2\right) + \oint_S \vec{S}\cdot d\vec{a} + \int_V dv\, \vec{E}\cdot\vec{J}_f = 0$

Ex: Current flowing through resistive wire of radius a



$\vec{E} = E e_z$ power dissipated in segment of length L: $|P| = IV = IEL$

$P = -IEL$

Magnetic field at wire surface

$$\vec{B} = \frac{4\pi}{c} I \cdot \frac{1}{2\pi a}\hat{e}_\theta = \frac{2I}{ca}\hat{e}_\theta$$

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = -\frac{IE}{2\pi a}\hat{e}_r$$

EM energy flux into segment of length L

$\oint_S \vec{S}\cdot d\vec{a} = \frac{IE}{2\pi a}2\pi a L = IEL$ This is the power dissipated

So Pointing's thm works!

2) Special Cases: Energy from a static charge distribution

a) $\vec{B} = 0$ no magnetic contribution to $\mathcal{E}$ no flow of energy so $\vec{S} = 0$

b) total electrostatic energy

$$U_e = \frac{1}{8\pi} \int_{V = \text{all of space}} dV\, \vec{D}\cdot\vec{E}$$

$$= -\frac{1}{8\pi} \int dV\, \vec{D}\cdot(\nabla\Phi)$$

$$= \frac{1}{8\pi} \int dV\, \Phi(\vec{\nabla}\cdot\vec{D})$$

$$= \frac{1}{2} \int dV\, \rho_f \Phi$$

used

$\vec{D}(\vec{\nabla}\Phi) = \nabla(\Phi D) - \Phi \vec{\nabla}\cdot\vec{D}$

& the fact that

$\int_V \nabla(\Phi D)\, dV = \oint_{S \text{ at infinity}} \Phi \vec{D}\cdot d\vec{a} = 0$

since $da \sim r^2$ $\Phi \sim \frac{1}{r}$ $D \sim \frac{1}{r^2}$

integral vanishes as $\frac{1}{r}$

notes: A. Bound charges still contribute to U_e since they modify Φ

B. $\rho_f \Phi$ is <u>not</u> an "energy density" we derived an expression for the total energy & then performed a partial integration over all of space & left out the boundary terms

The energy density is $\mathcal{E} = \frac{1}{8\pi}\left(\epsilon E^2 + \frac{1}{\mu} B^2\right)$

C. factor of $\frac{1}{2}$ is best understood by looking at the expression for Φ without bound charges

$$\Phi = \frac{dV'\, \rho(r')}{|\vec{r}-\vec{r}'|}$$

$$U_e = \frac{1}{2} \int dr\, dr'\, \frac{\rho(\vec{r})\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

each pair of points $\vec{r}, \vec{r}'$ appears twice. once as $\vec{r}, \vec{r}'$ & once as $\vec{r}', \vec{r}$

factor of $\frac{1}{2}$ solves this

Skipped discussion of $\Phi_\alpha(\vec{r}_\beta) = P_{\beta\alpha} q_\alpha$

where $P_{\alpha\beta} \equiv$ coeff of potential & depends on geometrical arrangement of conductors but not their charges