Maxwell Stress Tensor

0) Electric & Magnetic field posess energy

Conservation law:

$$\frac{d}{dt}\int_V dv\,\mathcal{E} + \oint_S \vec{S}\cdot d\vec{a} + \int_V dV\,\vec{E}\cdot\vec{J}_f = 0$$

$\mathcal{E} \equiv$ energy density, $\vec{S}$: energy flux per unit time & area

alternatives: $\vec{S}$ is "Power per cross sectional area"

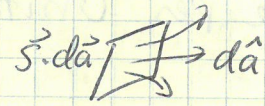
$\vec{S}$ is energy current density

We find, by rewriting maxwell's eqs.

$$\mathcal{E} = \frac{1}{8\pi}\left(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}\right), \quad \vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{H}$$

$\vec{S}$ is a flux so it has a direction

$\vec{S}\cdot d\vec{a}$ gives total energy transported across a unit area da perpendicular to $d\vec{a}$ per unit time



→ 1) Momentum in $\vec{E}, \vec{B}$

a) First, what form does the equation for conservation of momentum have in a continuous medium

* Need Momentum density

$\vec{g}$ where g_i = density of ith momentum component

* Momentum Flux is a tensor

$$(-\underline{\underline{T}})\cdot d\vec{a} = \sum_{ij}\left((-T_{ij})da_j\right)\hat{e}_i$$ is momentum transported across area da per unit time

$(-\underline{\underline{T}})d\vec{a}$ is a vector since momentum is a vector

Tensor $(-T_{ij})$: First index i refers to momentum being transported

Second index j refers to direction momentum is being transported in

Transport of momentum = exerting a force

Momentum transported out of an object per unit area time

= − Force exerted on object per unit area

= − Stress

Stress is a tensor $\underline{\underline{T}}$

Force is exerted by field itself. Diagonal elements of T_{ij} where $i = j$ is the force $\perp$ to the area $da \Rightarrow$ pressure

Off diagonal elements $i \neq j$: force // to $da \Rightarrow$ shear stress

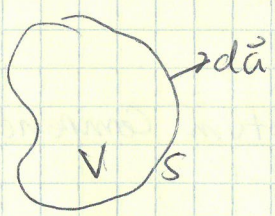
in equilibrium pressure is non zero

* For completeness we need a force exerted on object (volume V) by other entities than the fields (charges, currents)

Force per unit volume $= -\frac{\partial}{\partial t} g_{matter}$ where g_{matter} is momentum density of charges etc.

$\Rightarrow$ Look for relation of the form:

$$\frac{d}{dt}\int_V dV\, \vec{g}_{field} - \underbrace{\oint_S \underline{\underline{T}} \cdot d\vec{a}}_{\text{total flux of momentum out of V or total force exerted on V by EM field}} + \frac{d}{dt}\int_V dV\, \vec{g}_{matter} = 0$$



or locally: $$\frac{\partial \vec{g}_{field}}{\partial t} + \frac{\partial}{\partial t}\vec{g}_{matter} - \underbrace{\vec{\nabla} \cdot \underline{\underline{T}}}_{\sum_j \frac{\partial T_{ij}}{\partial x_j}\hat{e}_i} = 0$$

b) Lets derive this type of relation from the Maxwell eqs.

assume that there is no dielectric medium

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial B}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c}\vec{J}_f$$

Start w/ expression for Lorentz force density

$$\frac{\partial \vec{g}_{matter}}{\partial t} = f = \rho E + \frac{1}{c}\vec{J}\times\vec{B} = \frac{1}{4\pi}\left\{(\vec{\nabla}\cdot\vec{E})\vec{E} + \left(\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B}\right\}$$

$$A\times(B\times C) = -(B\times C)\times A$$

$$= \frac{1}{4\pi}\left\{\left[(\vec{\nabla}\cdot\vec{E})\vec{E} + \overbrace{(\vec{\nabla}\cdot\vec{B})}^{=0}\vec{B}\right] - \vec{B}\times(\vec{\nabla}\times\vec{B}) - \frac{1}{c}\left(\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B} + E\times\underbrace{(\nabla\times E)}_{-\frac{1}{c}\frac{\partial B}{\partial t}} - E\times(\nabla\times E)\right\}$$

$$= \frac{1}{4\pi}\left\{\underbrace{(\vec{\nabla}\cdot\vec{E})\vec{E} - \vec{E}\times(\vec{\nabla}\times\vec{E})}_{\text{lets look at } i^{th} \text{ component of this term}} + (\vec{\nabla}\cdot\vec{B})\vec{B} - B\times(\nabla\times B)\right\} - \frac{1}{4\pi c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

$$\frac{1}{4\pi}\{\cdots\}_i = \frac{1}{4\pi}\left\{\left(\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3}\right)E_i + E_1\frac{\partial E_i}{\partial x_1} + E_2\frac{\partial E_i}{\partial x_2} + E_3\frac{\partial E_i}{\partial x_3} - E_1\frac{\partial E_1}{\partial x_i} - E_2\frac{\partial E_2}{\partial x_i} - E_3\frac{\partial E_3}{\partial x_i}\right.$$

$$= \frac{1}{4\pi}\left\{\frac{\partial}{\partial x_1}(E_1E_i) + \frac{\partial}{\partial x_2}(E_2E_i) + \frac{\partial}{\partial x_3}(E_3E_i) - \frac{1}{2}\frac{\partial}{\partial x_i}E^2\right\}$$

Since E & B are symmetric in this formula we get a similar relation for B

$$\text{So } f = \underbrace{\frac{1}{4\pi}\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(E_jE_i - \frac{1}{2}E^2\delta_{ij} + \frac{\partial}{\partial x_j}B_jB_i - \frac{1}{2}B^2\delta_{ij}\right)}_{\equiv\ \vec{\nabla}\cdot\vec{T}_i\ =\ (\vec{\nabla}\cdot\underline{\underline{T}})_i} - \underbrace{\frac{1}{4\pi c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B})}_{-\frac{\partial}{\partial t}\frac{\vec{S}}{c^2}}$$

$$T_{ij} = \frac{1}{4\pi}\left(E_iE_j - \frac{1}{2}E^2\delta_{ij} + B_iB_j - \frac{1}{2}B^2\delta_{ij}\right) \quad \text{maxwell stress tensor}$$

$$\frac{\partial\vec{g}_{matter}}{\partial t} + \frac{\partial}{\partial t}\frac{\vec{S}}{c^2} - \vec{\nabla}\cdot\underline{\underline{T}} = 0 \qquad \text{where } \frac{\vec{S}}{c^2} = \vec{g}_{field} = \frac{1}{4\pi c}\vec{E}\times\vec{B}$$

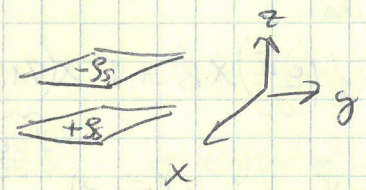
Ex: Let $\vec{E}$ be in the z direction

$$\underline{\underline{T}} = \underline{\underline{T}}_E + \underline{\underline{T}}_B$$

$$\underline{\underline{T}}_E = \begin{pmatrix} -\frac{1}{8\pi}E^2 & 0 & 0 \\ 0 & -\frac{1}{8\pi}E^2 & 0 \\ 0 & 0 & \frac{1}{8\pi}E^2 \end{pmatrix}$$

T_B would be similar if $\vec{B}$ in z direction

c) Ex: Parallel Capacitor Plates, surface charge density $\pm \rho_s$



$$\vec{E} = 4\pi\rho_s \hat{e}_z \qquad \underline{\underline{T}} = \frac{1}{8\pi}\begin{pmatrix} -E^2 & & \\ & E^2 & \\ & & E^2 \end{pmatrix}$$

force per unit area $d\vec{a} = da\,\hat{e}_z$

$= \frac{1}{8\pi}E^2\hat{e}_z$ attractive pressure $2\pi\rho_s^2$

note that pressure in x, y directions $\perp$ to field line is repulsive

note: $\oint_S \underline{\underline{T}}\cdot d\vec{a} = \frac{1}{8\pi}E^2\hat{e}_z da$ = force from top of plate we are ignoring the E field below the plate

<u>Summary:</u> ~~Pressure $= \frac{E^2}{8\pi} = 2\pi\rho_s^2$~~

Conservation laws for energy density & momentum density of EM field

$$\frac{d}{dt}\int_V dv\,\mathcal{E} + \oint_S \vec{S}\cdot d\vec{a} + \int_V dv\,\vec{E}\cdot\vec{J}_f = 0$$

$$\frac{d}{dt}\int_V dv\left(\vec{g}_{field} + \vec{g}_{matter}\right) - \underbrace{\oint_S \underline{\underline{T}}\cdot d\vec{a}}_{\text{total force exerted on V by EM Field}} = 0$$

$\mathcal{E} = \frac{1}{8\pi}\left(\epsilon E^2 + \frac{1}{\mu}B^2\right)$ energy density

$\vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{H})$ Poynting Vector

$\vec{g}_{field} = \frac{1}{c^2}\vec{S} = \frac{1}{4\pi c}\vec{E}\times\vec{H}$ momentum density

EM waves

Maxwell's eq w/o free charges, currents:

$$\vec{\nabla}\cdot\vec{E}=0 \qquad \vec{\nabla}\cdot\vec{D}=0 \qquad \vec{D}=\epsilon\vec{E}$$

$$\vec{\nabla}\cdot\vec{B}=0 \qquad \vec{B}=\mu\vec{H}$$

$$\vec{\nabla}\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{H}=\frac{1}{c}\frac{\partial\vec{D}}{\partial t}$$

$$\vec{\nabla}\times\vec{B}=\frac{\epsilon\mu}{c}\frac{\partial E}{\partial t}$$

Take curl of 3rd eq

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=-\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

vector identity

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E}=-\frac{\epsilon\mu}{c^2}\frac{\partial^2 E}{\partial t^2}$$

$$\nabla^2\vec{E}=+\frac{\epsilon\mu}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$$

Similarly $\nabla^2\vec{B}=\frac{\epsilon\mu}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$

Solutions for each component of $\vec{E},\vec{B}$ are solutions of the wave eq

$$\nabla^2\psi=\frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \quad w/ \quad \underbrace{v=\frac{c}{\sqrt{\epsilon\mu}}\equiv\frac{c}{n}}_{\text{Propagation velocity}}$$

n is the index of refrac.

$$\psi(r,t)=\sum_k A_k e^{i(k\cdot r-\omega t)}$$

taking the real part is implicit

"Fourier decomposition" of general solution of wave eq.

$$\omega=|\vec{k}|v=|\vec{k}|\frac{c}{n}$$

each term in the Fourier decomposition is called a "Plane wave"

$\vec{k}$: propagation vector

ω: angular freq.

Since Maxwell equations are linear, each term in Fourier decomposition must satisfy the Maxwell eqs. separately

b) For Single Plane wave Component:

$$\vec{E}(\vec{r},t) = \mathrm{Re}\ \vec{E}_{0\vec{k}}\, e^{i(k\cdot r - \omega t)}$$
$$\vec{B}(\vec{r},t) = \underbrace{\mathrm{Re}}\ \vec{B}_{0\vec{k}}\, e^{i(k\cdot r - \omega t)}$$

derivatives simply
Bring down a k or ω

taking real part
will usually be
left implicit

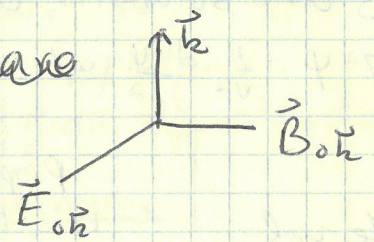
$$\vec{\nabla}\cdot\vec{E} = 0 \quad \Rightarrow \quad \vec{k}\cdot\vec{E} = 0$$
$$\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad \vec{k}\cdot\vec{B} = 0$$

Then:
$$\left.\begin{aligned} \vec{k}\cdot\vec{E}_{0\vec{k}} &= 0 \\ \vec{k}\cdot\vec{B}_{0\vec{k}} &= 0 \end{aligned}\right\} \vec{E},\vec{B} \perp \text{ to propagation direction } \vec{k}$$

$\nabla\times E + \frac{1}{c}\partial B/\partial t$ & $\vec{\nabla}\times\vec{B} - \frac{n^2}{c}\frac{\partial E}{\partial t}$ give

$$\left.\begin{aligned} \vec{k}\times\vec{E}_{0\vec{k}} &= \frac{\omega}{c}\vec{B}_{0k} \\ \vec{k}\times\vec{B}_{0\vec{k}} &= -n^2\frac{\omega}{c}\vec{E}_{0k} \end{aligned}\right\} \vec{E} \text{ \& } \vec{B} \perp \text{ to each other}$$

~~E & B are~~



$$\vec{B}_{0\vec{k}} = n\,\hat{e}_z \times \vec{E}_{0\vec{k}}$$
$$|\vec{B}_{0\vec{k}}| = n\,|\vec{E}_{0k}|$$

Note these results only apply if there is no free current!

Sometimes one compares $\vec{H}$ & $\vec{E}$

$$\vec{H}_{0\vec{k}} = \sqrt{\frac{\epsilon}{\mu}}\,\hat{e}_{\vec{k}} \times \vec{E}_{0\vec{k}} = \frac{1}{\eta}\,\hat{e}_k \times \vec{E}_{0\vec{k}} \qquad \eta = \sqrt{\frac{\mu}{\epsilon}} \text{ wave impedence}$$

Caveats: μ, ϵ may depend on ω

Then wave speed v depends on ω

<u>dispersion</u>: Shape of wave may change in time if more than one frequency is present

"group velocity" $\frac{d\omega}{dk}$ is speed at which information travels

nonlinear Media

$\mu_0 \epsilon$ may depend on $\vec{E}, \vec{B}$:

This leads to generation of waves at sum & difference frequencies (freq. doubling, freq doubling x talk used in Lasers)

[Aside: Simple origin μ, ϵ depend on $\vec{E} \Rightarrow n$ depends on $\vec{E}$

Take $n = n_0 + n_1 E$ $\quad E \propto \cos(\omega t)$

then $B \propto nE \propto n_0 \cos \omega t + n_1 \cos^2(\omega t)$

$$\cos^2(\omega t) = \frac{1}{2}\cos(\underbrace{2\omega t}_{\text{freq doubling}} + 1)$$

doubled freq leads feed back into $\vec{E}$ & so on.

d) Polarization

Looks at Plane wave propagating in the z direction

$\vec{k} = k\hat{e}_z \quad \vec{E} \perp \hat{e}_z \Rightarrow \vec{E}$ in x-y plane

$$\vec{E}(\vec{r},t) = (E_1 \hat{e}_1 + E_2 \hat{e}_2)\, e^{i(kz-\omega t)}$$

$\hat{e}_1, \hat{e}_2$ are called "Polarization vectors"

E_1, E_2 are complex numbers

$E_1 = E_{10} e^{i\alpha}, \; E_2 = E_{20} e^{i\beta}$

Remember we only care about real parts

$$\vec{E}(\vec{r},t) = E_{10}\hat{e}_1 \cos(kz - \omega t + \alpha) + E_{20}\hat{e}_2 \cos(kz - \omega t + \beta)$$

* If $\alpha = \beta$ or $\alpha = \beta + \pi \pmod{2\pi}$ then

$$\vec{E}(\vec{r},t) = (E_{10}\hat{e}_1 \pm E_{20}\hat{e}_2)\cos(kz - \omega t + \alpha)$$

"linearly polarized wave"

Since direction of $\vec{E}$ is always along

$$E_{10}\hat{e}_1 + E_{20}\hat{e}_2$$

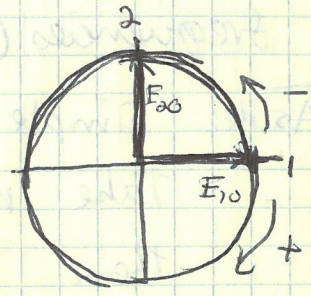
* If $\beta = \alpha \pm \frac{\pi}{2}$ & $E_{10} = E_{20}$

$$\vec{E}(\vec{r},t) = E_{10}\left(\hat{e}_1 \cos(kz - \omega t + \alpha) \mp \hat{e}_2 \sin(kz - \omega t + \alpha)\right)$$

"Circularly polarized"

$\vec{E}$ rotates in $\hat{e}_1, \hat{e}_2$ plane

more generally ellipsoidal trace of $\vec{E}$ in $\hat{e}_1, \hat{e}_2$ plane



note one distinguishes "left circularly" & "right circularly" polarized light

e) Pointing vector points in propagation direction

Complex number convention

$\vec{E}, \vec{B}$ are written as real parts of complex quantities

This is useful for Fourier decomposition since

$$\mathrm{Re}[\text{Summation}] = \text{Sum}[\text{real parts}]$$

Products however are tricky

$$\mathrm{Re}[a b] \neq \mathrm{Re}[a]\,\mathrm{Re}[b]$$

** Must take real parts before multiplying

If one is interested in a <u>time average only</u> there is a trick involving complex #'s.

$$F(t) = \sum_j \mathrm{Re}\, F_j e^{i\omega_j t} \qquad F_j = F_{0j} e^{i\alpha}$$

$$G(t) = \sum_j \mathrm{Re}\, G_j e^{i\omega_j t}$$

what is $\langle F(t) G(t) \rangle_{\text{time avg}}$?

write $F(t) = \frac{1}{2} \sum_j \left(F_j e^{i\omega_j t} + F_j^* e^{-i\omega_j t} \right)$

$$G(t) = \frac{1}{2} \sum_j \left(G_j e^{i\omega_j t} + G_j^* e^{-i\omega_j t} \right)$$

assume all ω_j are > 0

$$\Rightarrow F(t)G(t) = \frac{1}{4} \sum_{jk} \Big(F_j G_k e^{i(\omega_j + \omega_k)t} + F_j^* G_k e^{i(\omega_k - \omega_j)t}$$
$$+ F_j G_k^* e^{i(\omega_j - \omega_k)t} + F_j^* G_k^* e^{-i(\omega_k + \omega_j)t} \Big)$$

take time average ~~[illegible]~~ $\to 0$ if $\omega_j = \omega_k$ since $\langle \rangle_t \sin, \cos = 0$

$$\langle F(t)G(t) \rangle = \frac{1}{4} \sum_j \left(F_j G_j^* + F_j^* G_j \right) = \frac{1}{2} \sum_j \mathrm{Re}\, F_j G_j^*$$

☆ Example: time average of Poynting vector

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{H}$$

$$\left.\begin{aligned} \vec{E} &= \vec{E}_0 e^{i(kz - \omega t)} \\ \vec{H} &= \vec{H}_0 e^{i(kz - \omega t)} \end{aligned}\right\} \text{ z coord along propagation direction}$$

$$\vec{H}_0 = \frac{1}{\mu} \vec{B}_0 = \underbrace{\sqrt{\frac{\epsilon}{\mu}}}_{\frac{1}{\eta} \text{ wave impedence}} \hat{e}_{\vec{k}} \times \vec{E}_0 \quad \text{in dielectric medium/vacuum}$$

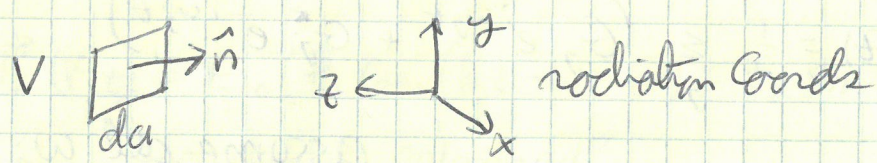
$$\langle \vec{S} \rangle = \frac{c}{4\pi} \frac{1}{2} \mathrm{Re}\, \vec{E}_0 \times \vec{H}_0^*$$

$$= \frac{c}{4\pi} \frac{1}{2} \mathrm{Re}\, \vec{E}_0^* \times \vec{H}_0$$

$$= \frac{c}{8\pi\eta} |E_0|^2 \hat{e}_k$$

Similarly energy density: $\mathcal{E} = \frac{1}{8\pi} \frac{1}{2} \left(\epsilon |E_0|^2 + \mu |H_0|^2 \right)$

3) Radiation pressure



area element da:

force ⊥ area exerted by radiation

$\vec{F} = \hat{n} \cdot \underline{\underline{T}}\, \hat{n}\, da$ This is the force that pulls on area because it is along outward normal

Pressure $P = -\hat{n} \cdot \underline{\underline{T}} \cdot \hat{n}$

For wave traveling in z direction the pressure on a surface in the xy plane is $-T_{zz}$

If we choose x along $\vec{E}$, y along $\vec{B}$

$\vec{E}_x$ so x positive is positive

$\vec{B}_y$ so y positive is positive

$$T = \frac{1}{8\pi}\begin{pmatrix} E_x^2 - B_y^2 & & \\ & -E_x^2 + B_y^2 & \\ & & -E_x^2 - B_y^2 \end{pmatrix}$$

$$P = \frac{1}{8\pi}\left(E_x^2 + B_y^2\right)$$

$$\langle P \rangle = \frac{1}{16\pi}\left(|E_0|^2 + |B_0|^2\right) = \frac{1}{8\pi}|E_0|^2$$ in vacuum where $\mu = 1$

Since $|\vec{B}_{0k}| = n|\vec{E}_{0k}|$

alternative method: Calculate momentum density of EM field $\vec{g} = \frac{1}{c^2}\vec{S}$ carried at velocity $c\hat{e}_z$

$$\langle P \rangle = c\langle g_z \rangle = \frac{c}{8\pi}|E_0|^2$$

Waves in a Conductor

a) Look for plane wave solutions at angular frequency ω

$$\vec{\nabla}\cdot\vec{E}=0$$
$$\vec{\nabla}\cdot\vec{B}=0$$
$$\vec{\nabla}\times\vec{E}+\frac{1}{c}\frac{\partial B}{\partial t}=0$$
$$\vec{\nabla}\times\vec{B}-\frac{\epsilon\mu}{c}\frac{\partial E}{\partial t}=\frac{4\pi\mu}{c}J$$
$$=\frac{4\pi\mu\sigma}{c}\vec{E}$$

Conductor $\rho=0$

But $\vec{J}=\sigma\vec{E}\neq 0$ if $\vec{E}\neq 0$

Proceed as before when we looked at EM waves in an insulating medium

$$\nabla^2\vec{E}-\frac{4\pi\sigma\mu}{c^2}\frac{\partial\vec{E}}{\partial t}-\frac{\epsilon\mu}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}=0$$

$$\nabla^2\vec{B}-\underbrace{\frac{4\pi\sigma\mu}{c^2}\frac{\partial B}{\partial t}}_{\text{new terms}}-\frac{\epsilon\mu}{c^2}\frac{\partial^2 E}{\partial t^2}$$

an satz: $\vec{E}=\vec{E}_0\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\rightarrow\vec{E}_0\,e^{i(kz-\omega t)}$

$\vec{B}=\vec{B}_0\,e^{i(k\cdot r-\omega t)}\rightarrow\vec{B}_0\,e^{i(kz-\omega t)}$

where z is the measure of the $\perp$ distance of a wave front from the origin

$$k^2-i\frac{4\pi\sigma\mu\omega}{c^2}-\frac{\epsilon\mu\omega^2}{c^2}=0$$

$$k^2=\frac{\epsilon\mu\omega^2}{c^2}\left(1+\frac{4\pi i\sigma}{\epsilon\omega}\right)$$

$$k=\pm\frac{\omega}{c}\sqrt{\epsilon\mu}\sqrt{1+\frac{4\pi i\sigma}{\epsilon\omega}}$$

where $\sqrt{i}=\frac{i+1}{\sqrt{2}}$

What does it mean to have a complex k?

$k = \alpha + i\beta$

$$E(\vec{r},t) = \vec{E}_0 \, e^{i(\alpha z - \omega t) - \beta z}$$

$\alpha = Re(k)$ propagation $\qquad \beta = Im(k)$ exponential decay

need to choose "+" sign in expression for k otherwise solution diverges (not physical)

Alternative representation:

write eq for k at fixed ω as:

$$k^2 = \frac{\epsilon(\omega)\mu\omega^2}{c^2} \qquad \text{where } \epsilon(\omega) = \epsilon + \frac{4\pi i \sigma}{\omega}$$

Same form as eq for k in an insulating dielectric medium but with a "complex" dielectric constant

$k = \frac{\omega}{c}\sqrt{\mu\epsilon(\omega)}$ also defines complex index of refraction

$$n(\omega) = \sqrt{\epsilon\mu}\left(1 + \frac{4\pi i \sigma}{\epsilon\omega}\right)^{1/2} \qquad n(\omega) = \sqrt{\epsilon(\omega)\mu} \Rightarrow k = \frac{\omega}{c} n(\omega)$$

So it all looks the same as before only w/ complex #'s.

b) Relation between complex field amplitudes:

$\vec{B}_0 = n\,\hat{e}_k \times \vec{E}_0$ is n is complex $\vec{E}$ & $\vec{B}$ are no longer in phase

- Phase that $\vec{B}$ lags behind $\vec{E}$ is the phase of $n(\omega)$
- magnitude of $\vec{B}$ & $\vec{E}$ relates to magnitude of n

$$|n(\omega)| = \frac{|\vec{B}_0|}{|\vec{E}_0|} = \sqrt{\epsilon\mu}\left(1 + \left(\frac{4\pi\sigma}{\epsilon\omega}\right)^2\right)^{1/4}$$