Read all of the following information before starting the exam:

- Put your name on the exam now.
- Show all work, clearly and in order, if you want to get full credit.
- Circle or otherwise indicate your final answers.
- The first problem is more conceptual, the next two are more computational.
- Question 3 (c) is a bonus question worth 10 points. The total exam score cannot exceed 100, but the bonus question can help you make up points lost elsewhere.

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- It is your responsibility to make sure that you have all of the pages!
- Good luck!
Question 1: Rectangular Waveguide [50 Points]

Consider the propagation of TE waves in a rectangular waveguide with \( a > b \):

All field components have the \( z \) and \( t \) dependence \( e^{i(k_g z - \omega t)} \) and can be described in terms of \( B^0_z \), which satisfies Helmholtz’ equation

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2 \right) B^0_z = 0.
\]  

(1)

Together with the boundary conditions \( \partial B^0_z / \partial n |_S = 0 \), this gives the following solution:

\[
B^0_z = B^0 \cos \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi y}{b} \right),
\]  

(2)

where \( m, n \) are positive integers, at least one of which must be nonzero.

a) [10 pts] Write down an expression for the cut-off frequency \( \omega_c \equiv \omega_{mn} \).
b) [10 pts] Consider the mode TE_{34}.

(i) Calculate the corresponding cutoff frequency $\omega_{mn}$.

For each of the following cases, does the mode propagate or not? If not, what is the physical reason?

(ii) $\omega > \omega_c$

(iii) $\omega < \omega_c$
c) [20 pts] How should I pick $a/b$ to maximize the range of frequencies over which *only* TE$_{10}$ propagates? What is this maximum range? Make sure you explain your reasoning.

(This is called “maximizing the single-mode bandwidth” and is desirable for many applications.)
d) [10 pts] Using $v_g = \frac{\partial \omega}{\partial k}$, show that the group velocity of a TE$_{mn}$ mode is

$$v_g = \frac{c}{\omega} \sqrt{\omega^2 - \omega_{mn}^2} < c.$$  \hspace{1cm} (3)
Question 2: Energy Traveling in Rectangular Waveguide [30 Points]

Consider a TE_{10} mode traveling down the rectangular waveguide of Question 1, with $B_z^0$ given by eqn. (2).

**Hints:**

Throughout this question, you may find the following useful:

\[
\int_0^a \sin^2\left(\frac{m\pi x}{a}\right) \, dx = \int_0^a \cos^2\left(\frac{m\pi x}{a}\right) = \frac{a}{2}
\]

You may also want to make use of the time-average product theorem:

\[
\langle F \cdot G \rangle \rightarrow \frac{1}{2} F_0 \cdot G_0^* = \frac{1}{2} F_0^* \cdot G_0,
\]

where $F_0, G_0$ are complex amplitudes and the multiplication could be any kind, including vector cross- and dot-product.

— Questions on the next pages. —
a) [5 pts] Recall that we derived in lecture how the longitudinal field components in a waveguide determine all other components:

\[
\begin{align*}
E_x^0 &= \frac{i}{k_c^2} \left( k_0 \frac{\partial B_z^0}{\partial y} + k_g \frac{\partial E_z^0}{\partial x} \right) \\
E_y^0 &= -\frac{i}{k_c^2} \left( k_0 \frac{\partial B_z^0}{\partial x} - k_g \frac{\partial E_z^0}{\partial y} \right) \\
B_x^0 &= -\frac{i}{k_c^2} \left( k_0 \frac{\partial E_z^0}{\partial y} - k_g \frac{\partial B_z^0}{\partial x} \right) \\
B_y^0 &= \frac{i}{k_c^2} \left( k_0 \frac{\partial E_z^0}{\partial x} + k_g \frac{\partial B_z^0}{\partial y} \right)
\end{align*}
\]

Use these relations to find all field components of the TE_{10} mode.
b) [5 pts] Show that

\[ \langle S \rangle_{10} = e\frac{c}{8\pi} \left( \frac{a}{\pi B_0} \right)^2 k_0 k_g \sin^2 \left( \frac{\pi x}{a} \right) \]

c) [5 pts] Calculate the total power \( P_{10} \) transmitted by the mode.
d) [5 pts] Show that the time-averaged energy-density of the electromagnetic fields of the TE_{10} mode is

\[ \langle \varepsilon \rangle_{10} = \frac{(B_0^2)^2}{16\pi^3} \left\{ (k_0^2 + k_0^2) a^2 \sin^2 \left( \frac{\pi x}{a} \right) + \cos^2 \left( \frac{\pi x}{a} \right) \right\} \]


e) [5 pts] Calculate the time-averaged energy-density per unit length along the waveguide of the mode.
f) [5 pts] Using your answers to parts (c) and (e), show that the energy in the TE_{10} mode travels at the group velocity (defined in part (d) of Question 1).
Scalar Invariants of Electromagnetic Fields [20 Points]

a) [10 pts] Recall that the dual field tensor $G_{\mu\nu}$ can be obtained from the electromagnetic field tensor $F_{\mu\nu}$ by replacing $E \rightarrow B$ and $B \rightarrow -E$. Write down $F_{\mu\nu}$ and $G_{\mu\nu}$. 
b) [10 pts] Show that $\mathbf{E} \cdot \mathbf{B}$ and $E^2 - B^2$ are invariant under Lorentz transformations.
c) **Bonus Question [10 pts]:** Show that $S^2 - c^2 \varepsilon^2$ is also Lorentz invariant, where $S$ is the magnitude of the Poynting vector and $\varepsilon$ is the energy density of the fields.