PHYS 327 PRELIM 2

Prof. Itai Cohen, Fall 2009

Monday, 11/23/09 **Name:**

Read all of the following information before starting the exam:

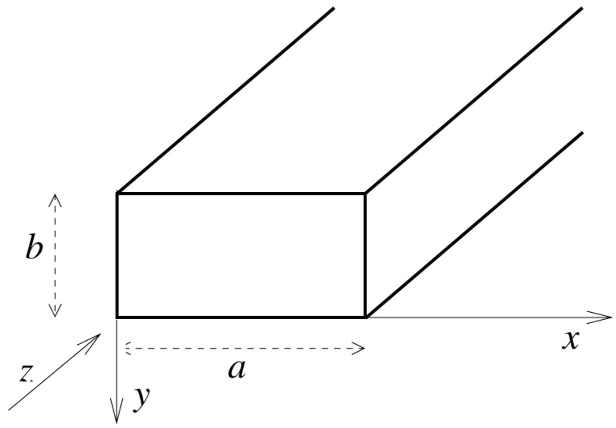
- Put your name on the exam **now**.
- Show all work, clearly and in order, if you want to get full credit.
- Circle or otherwise indicate your final answers.
- The first problem is more conceptual, the next two are more computational.
- Question 3 (c) is a bonus question worth 10 points. The total exam score cannot exceed 100, but the bonus question can help you make up points lost elsewhere.

Problem #	Score
1	/50
2	/30
3	/20
3(c)	/10
Total	/100

- It is your responsibility to make sure that you have all of the pages!
- Good luck!

Question 1: Rectangular Waveguide [50 Points]

Consider the propagation of TE waves in a rectangular waveguide with $a > b$:



All field components have the z and t dependence $e^{i(k_g z - \omega t)}$ and can be described in terms of B_z^0, which satisfies Helmholz' equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2 \right) B_z^0 = 0. \tag{1}$$

Together with the boundary conditions $\partial B_z^0 / \partial n|_S = 0$, this gives the following solution:

$$B_z^0 = B^0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right), \tag{2}$$

where m, n are positive integers, at least one of which must be nonzero.

a) [10 pts] Write down an expression for the cut-off frequency $\omega_c \equiv \omega_{mn}$.

b) [10 pts] Consider the mode TE_{34}.

(i) Calculate the corresponding cutoff frequency ω_{mn}.

For each of the following cases, does the mode propagate or not? If not, what is the physical reason?

(ii) $\omega > \omega_c$

(iii) $\omega < \omega_c$

c) [20 pts] How should I pick a/b to maximize the range of frequencies over which *only* TE_{10} propagates? What is this maximum range? Make sure you explain your reasoning.

(This is called "maximizing the single-mode bandwidth" and is desirable for many applications.)

d) [10 pts] Using $v_g = \frac{\partial \omega}{\partial k_g}$, show that the group velocity of a TE_{mn} mode is

$$v_g = \frac{c}{\omega}\sqrt{\omega^2 - \omega_{mn}^2} < c. \tag{3}$$

Question 2: Energy Traveling in Rectangular Waveguide [30 Points]

Consider a TE_{10} mode traveling down the rectangular waveguide of Question 1, with B_z^0 given by eqn. (2).

Hints:
Throughout this question, you may find the following useful:

$$\int_0^a \sin^2\left(\frac{m\pi x}{a}\right) dx = \int_0^a \cos^2\left(\frac{m\pi x}{a}\right) = \frac{a}{2}$$

You may also want to make use of the time-average product theorem:

$$\langle F \cdot G \rangle \rightarrow \frac{1}{2} F_0 \cdot G_0^* = \frac{1}{2} F_0^* \cdot G_0,$$

where F_0, G_0 *are complex amplitudes and the multiplication could be any kind, including vector cross- and dot-product.*

— Questions on the next pages. —

a) [5 pts] Recall that we derived in lecture how the longitudinal field components in a waveguide determine all other components:

$$
\begin{aligned}
E_x^0 &= \frac{i}{k_c^2}\left(k_0\frac{\partial B_z^0}{\partial y} + k_g\frac{\partial E_z^0}{\partial x}\right)\\
E_y^0 &= -\frac{i}{k_c^2}\left(k_0\frac{\partial B_z^0}{\partial x} - k_g\frac{\partial E_z^0}{\partial y}\right)\\
B_x^0 &= -\frac{i}{k_c^2}\left(k_0\frac{\partial E_z^0}{\partial y} - k_g\frac{\partial B_z^0}{\partial x}\right)\\
B_y^0 &= \frac{i}{k_c^2}\left(k_0\frac{\partial E_z^0}{\partial x} + k_g\frac{\partial B_z^0}{\partial y}\right)
\end{aligned}
$$

Use these relations to find all field components of the TE_{10} mode.

b) [5 pts] Show that

$$\langle S\rangle_{10} = \mathbf{e}_z \frac{c}{8\pi}\left(\frac{a}{\pi}B_0\right)^2 k_0 k_g \sin^2\left(\frac{\pi x}{a}\right)$$

c) [5 pts] Calculate the total power P_{10} transmitted by the mode.

d) [5 pts] Show that the time-averaged energy-density of the electromagnetic fields of the TE_{10} mode is

$$\langle \varepsilon \rangle_{10} = \frac{(B^0)^2}{16\pi^3} \left\{ (k_0^2 + k_g^2) a^2 \sin^2\left(\frac{\pi x}{a}\right) + \cos^2\left(\frac{\pi x}{a}\right) \right\}$$

e) [5 pts] Calculate the time-averaged *energy-density per unit length along the waveguide* of the mode.

f) [5 pts] Using your answers to parts (c) and (e), show that the energy in the TE_{10} mode travels at the group velocity (defined in part (d) of Question 1).

Scalar Invariants of Electromagnetic Fields [20 Points]

a) [10 pts] Recall that the dual field tensor $G_{\mu\nu}$ can be obtained from the electromagnetic field tensor $F_{\mu\nu}$ by replacing $\mathbf{E} \to \mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}$. Write down $F_{\mu\nu}$ and $G_{\mu\nu}$.

b) [10 pts] Show that $\mathbf{E} \cdot \mathbf{B}$ and $E^2 - B^2$ are invariant under Lorentz transformations.

c) **Bonus Question [10 pts]:** Show that $S^2 - c^2\varepsilon^2$ is also Lorentz invariant, where S is the magnitude of the Poynting vector and ε is the energy density of the fields.