PHYS 327 PRELIM 2

Prof. Itai Cohen, Fall 2009

Monday, 11/23/09 **Name:** SOLUTIONS

Read all of the following information before starting the exam:

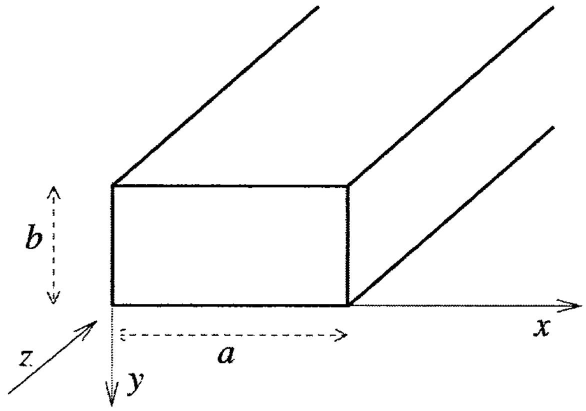
- Put your name on the exam **now**.
- Show all work, clearly and in order, if you want to get full credit.
- Circle or otherwise indicate your final answers.
- The first problem is more conceptual, the next two are more computational.
- Question 3 (c) is a bonus question worth 10 points. The total exam score cannot exceed 100, but the bonus question can help you make up points lost elsewhere.

Problem #	Score
1	/50
2	/30
3	/20
3(*c*)	/10
Total	/100

- It is your responsibility to make sure that you have all of the pages!
- Good luck!

Question 1: Rectangular Waveguide [50 Points]

Consider the propagation of TE waves in a rectangular waveguide with $a > b$:



All field components have the z and t dependence $e^{i(k_g z - \omega t)}$ and can be described in terms of B_z^0, which satisfies Helmholz' equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) B_z^0 = 0. \tag{1}$$

Together with the boundary conditions $\partial B_z^0/\partial n|_S = 0$, this gives the following solution:

$$B_z^0 = B^0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right), \tag{2}$$

where m, n are positive integers, at least one of which must be nonzero.

a) [10 pts] Write down an expression for the cut-off frequency ω_c. $\equiv \omega_{mn}$

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) B_z^0 = k_c^2 B_z^0$$

$$\Rightarrow -\left(-\left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right) = k_c^2 \Rightarrow k_c^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

$$k_c^2 = \frac{\omega_c^2}{c^2} \Rightarrow \omega_c = \pi c \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

b) [10 pts] Consider the mode TE_{34}.

(i) Calculate [redacted] the corresponding cutoff frequency ω_{mn}.

$$\omega_{34} = \pi c \sqrt{\frac{9}{a^2} + \frac{16}{b^2}}$$

For each of the following cases, does the mode propagate or not? If not, what is the physical reason?

(ii) $\omega > \omega_c$

(iii) $\omega < \omega_c$

(ii) $\omega > \omega_c$ propagates

(iii) $\omega < \omega_c$ does not propagate.

(Several good explanations)

$\rightarrow$ • We cannot find an angle of entry that would set up the required wavelengths along the x and y direction

• the group velocity becomes imaginary

• if $\omega < \omega_c$, $k_0 < k_c \Rightarrow k_g = \sqrt{k_0^2 - k_c^2}$ becomes imaginary

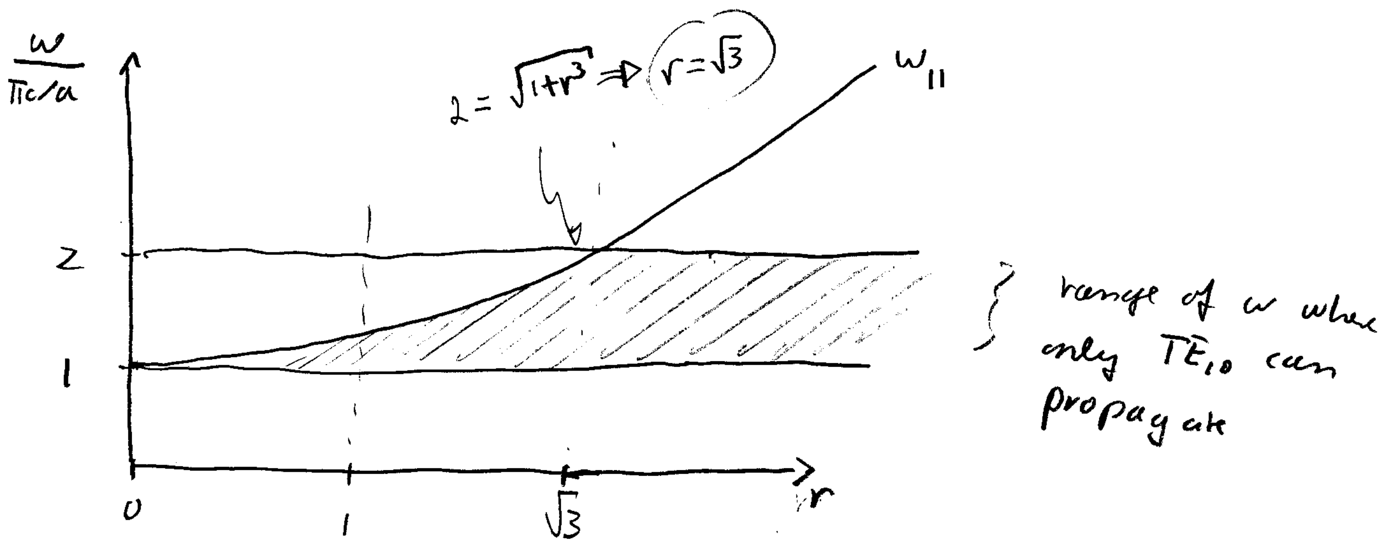
c) [20 pts] How should I pick a/b to maximize the range of frequencies over which *only* TE_{10} propagates? What is this maximum range? Make sure you explain your reasoning.

(This is called "maximizing the single-mode bandwidth" and is desirable for many applications.)

Let $r = \frac{a}{b} \Rightarrow \omega_{mn} = \frac{\pi c}{a}\sqrt{m^2 + n^2 r^2}$

As we increase ω, after TE_{10} the next mode to become available for propagation is either TE_{20} or TE_{11}:

$$\omega_{10} = \frac{\pi c}{a} \qquad \omega_{20} = \frac{\pi c}{a}\cdot 2 \qquad \omega_{11} = \frac{\pi c}{a}\sqrt{1+r^2}$$



$\Rightarrow$ Hence for $\frac{a}{b} > \sqrt{3}$, only TE will propagate for max range of

$$\frac{\pi c}{a} < \omega < 2\,\frac{\pi c}{a}$$

(Picking $\frac{a}{b} \approx 2$ is also an acceptable answer)

d) [10 pts] Using $v_g = \frac{\partial \omega}{\partial k_g}$, show that the group velocity of a TE_{mn} mode is

$$v_g = \frac{c}{\omega}\sqrt{\omega^2 - \omega_{mn}^2} < c. \tag{3}$$

$$\frac{\omega^2}{c^2} = k_0^2 = k_g^2 + k_c^2$$

$$\omega = c\sqrt{k_g^2 + k_c^2}$$

$$\frac{\partial \omega}{\partial k_g} = \frac{c\,k_g}{\sqrt{k_g^2 + k_c^2}} = \frac{c\sqrt{k_0^2 - k_c^2}}{\omega / c}$$

$$= \frac{c^2}{\omega}\frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}$$

$$\boxed{v_g = \frac{c}{\omega}\sqrt{\omega^2 - \omega_{mn}^2}}$$

Question 2: Energy Traveling in Rectangular Waveguide [30 Points]

Consider a TE_{10} mode traveling down the rectangular waveguide of Question 1, with B_z^0 given by eqn. (2).

Hints:
Throughout this question, you may find the following useful:

$$\int_0^a \sin^2\left(\frac{m\pi x}{a}\right) dx = \int_0^a \cos^2\left(\frac{m\pi x}{a}\right) = \frac{a}{2}$$

You may also want to make use of the time-average product theorem:

$$\langle F \cdot G \rangle \rightarrow \frac{1}{2} F_0 \cdot G_0^* = \frac{1}{2} F_0^* \cdot G_0,$$

where F_0, G_0 are complex amplitudes and the multiplication could be any kind, including vector cross- and dot-product.

— Questions on the next pages. —

a) [5 pts] Recall that we derived in lecture how the longitudinal field components in a waveguide determine all other components:

$$
\begin{aligned}
E_x^0 &= \frac{i}{k_c^2}\left(k_0\frac{\partial B_z^0}{\partial y} + k_g\frac{\partial E_z^0}{\partial x}\right)\\
E_y^0 &= -\frac{i}{k_c^2}\left(k_0\frac{\partial B_z^0}{\partial x} - k_g\frac{\partial E_z^0}{\partial y}\right)\\
B_x^0 &= -\frac{i}{k_c^2}\left(k_0\frac{\partial E_z^0}{\partial y} - k_g\frac{\partial B_z^0}{\partial x}\right)\\
B_y^0 &= \frac{i}{k_c^2}\left(k_0\frac{\partial E_z^0}{\partial x} + k_g\frac{\partial B_z^0}{\partial y}\right)
\end{aligned}
$$

Use these relations to find all field components of the TE_{10} mode.

$E_z^0 = 0$ $\quad$ | $\quad$ $\underline{B_z^0 = B^0 \cos\left(\frac{\pi x}{a}\right)}$ $\qquad$ $k_c = \frac{\pi}{a}$,

$\Rightarrow$ $E_x^0 = 0$;

$$E_y^0 = +\frac{ia^2}{\pi^2} k_0 B^0 \sin\left(\frac{\pi x}{a}\right)\frac{\pi}{a} = \underline{\frac{ia}{\pi} k_0 B^0 \sin\frac{\pi x}{a}}$$

$$B_x^0 = -\frac{ia^2}{\pi^2}(-1)\, k_g\, B^0(-1)\sin\frac{\pi x}{a}\,\frac{\pi}{a} = \underline{-\frac{ia}{\pi} k_g B^0 \sin\frac{\pi x}{a}}$$

$B_y^0 = 0$

b) [5 pts] Show that

$$\langle S \rangle_{10} = \mathbf{e}_z \frac{c}{8\pi} \left(\frac{a}{\pi} B_0 \right)^2 k_0 k_g \sin^2 \left(\frac{\pi x}{a} \right)$$

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{H} \longrightarrow \langle \vec{S} \rangle = \frac{c}{8\pi} \mathrm{Re} \left(\vec{E}_0 \times \vec{B}_0^* \right)$$

$$\langle \vec{S} \rangle = \frac{c}{8\pi} \mathrm{Re} \left(0, \frac{ia}{\pi} k_0 B^0 \sin \frac{\pi x}{a}, 0 \right) \times \left(\frac{ia}{\pi} k_g B^0 \sin \frac{\pi x}{a}, 0, B^0 \cos \frac{\pi x}{a} \right),$$

drops out, since imag part

$$= \frac{c}{8\pi} \frac{a^2}{\pi^2} (B^0)^2 \sin^2\left(\frac{\pi x}{a}\right) k_0 k_g \vec{e}_z$$

$$\Rightarrow \boxed{\langle \vec{S}_{10} \rangle = k_0 k_g \frac{c a^2}{8\pi^3} (B^0)^2 \sin^2\left(\frac{\pi x}{a}\right)}$$

c) [5 pts] Calculate the total time-ave. averaged power P_{10} transmitted by the mode.

$$|\vec{S}| = \frac{\text{Power}}{\text{area}} \Rightarrow \text{Power transmitted in } z \text{ direction} = \int (\text{cross-sec. area}) \cdot |\langle \vec{S}_z \rangle|$$

$$P_{10} = \int_0^a b \, dx \, \langle S_{10} \rangle = \frac{c}{16\pi^3} k_0 k_g a^3 b (B^0)^2$$

d) [5 pts] Show that the time-averaged energy-density of the electromagnetic fields of the TE_{10} mode is

$$\langle \varepsilon \rangle_{10} = \frac{(B^0)^2}{16\cancel{\pi^3}\,\pi}\left\{(k_0^2 + k_g^2)\underbrace{a^2}_{\pi^2}\sin^2\left(\frac{\pi x}{a}\right) + \cos^2\left(\frac{\pi x}{a}\right)\right\}$$

absolute value in both rectangular and $\mathbb{C}$-sense!

$$\mathcal{E} = \frac{1}{8\pi}\left(\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B}\right) \Rightarrow \langle \mathcal{E} \rangle = \frac{1}{16\pi}\left(|\vec{E}^0|^2 + |\vec{B}^0|^2\right)$$

$$\langle \varepsilon \rangle = \frac{1}{16\pi}\left(\sum E_i^0 E_i^{0*} + \sum B_i B_i^*\right)$$

$$= \frac{(B^0)^2}{16\pi}\left(\frac{k_0^2 a^2}{\pi^2}\sin^2\left(\frac{\pi x}{a}\right) + \cos^2\left(\frac{\pi x}{a}\right) + \frac{k_g^2 a^2}{\pi^2}\sin^2\left(\frac{\pi x}{a}\right)\right)$$

$$\boxed{\langle \varepsilon \rangle = \frac{B^{0\,2}}{16\pi}\left((k_0^2 + k_g^2)\frac{a^2}{\pi^2}\sin^2\left(\frac{\pi x}{a}\right) + \cos^2\left(\frac{\pi x}{a}\right)\right)}$$

e) [5 pts] Calculate the time-averaged *energy-density per unit length along the waveguide* of the mode.

$$\int \langle \varepsilon \rangle \cdot \underset{\text{cross sectional area}}{da} = \frac{B_0^2 ba}{32\pi}\left[\frac{k_0^2 a^2}{\pi^2} + \underset{\boxed{k_c^2 a^2/\pi^2}}{1} + \frac{k_g^2 a^2}{\pi^2}\right]$$

$$= \frac{(B^0)^2\, b a^3 k_0^2}{16\pi^3}$$

f) [5 pts] Using your answers to parts (c) and (e), show that the energy in the TE_{10} mode travels at the group velocity (defined in part (d) of Question 1).

$$\text{Power} = \frac{\text{energy}}{\text{time}} \qquad \int \langle \mathcal{E} \rangle \cdot da = \frac{\text{energy}}{\text{length}}$$

$$2 \quad \frac{\text{Power}}{\int \langle \mathcal{E} \rangle \cdot da} = \frac{\text{length}}{\text{time}} = \text{speed at which energy travels!}$$

$$v = \frac{c}{\cancel{16\pi^3}} k_0 k_g \cancel{a^3 b} \cancel{(B^0)^2} \times \frac{\cancel{16\pi^3}}{\cancel{(B^0)^2} \cancel{b} \cancel{a^3} k_0^2}$$

$$= \frac{c k_g}{k_0}$$

$$v_g = \frac{1}{k_0} \cdot \sqrt{c^2 k_0^2 - c^2 k_c^2} = \frac{c k_g}{k_0}$$

$$\boxed{v = v_g}$$

$\square$

Scalar Invariants of Electromagnetic Fields [20 Points]

a) [10 pts] Recall that the dual field tensor $G_{\mu\nu}$ can be obtained from the electromagnetic field tensor $F_{\mu\nu}$ by replacing $\mathbf{E} \to \mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}$. Write down $F_{\mu\nu}$ and $G_{\mu\nu}$.

$$F_{\mu\nu} = \begin{bmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{bmatrix}$$

$$G_{\mu\nu} = \begin{bmatrix} 0 & -E_3 & E_2 & -iB_1 \\ E_3 & 0 & -E_1 & -iB_2 \\ -E_2 & E_1 & 0 & -iB_3 \\ iB_1 & iB_2 & iB_3 & 0 \end{bmatrix}$$

b) [10 pts] Show that $\mathbf{E} \cdot \mathbf{B}$ and $E^2 - B^2$ are invariant under Lorentz transformations.

We will use the Lorentz invariants $Tr(F^2)$ and $Tr(GF)$.

- $$Tr\,F^2 = F_{ij} F_{ji} = -\sum_{i,j} (F_{ij}^2)$$

$$= B_1^2 + B_2^2 + B_3^2 - E_1^2 - E_2^2 - E_3^2 = -(|\vec{E}|^2 - |\vec{B}|^2)$$

$\Rightarrow$ $|\vec{E}|^2 - |\vec{B}|^2$ is a Lorentz invariant

$$\mathrm{Tr}(GF) = B_3E_3 + B_2E_2 + E_1B_1$$
$$+ E_3B_3 + E_1B_1 + E_2B_2$$
$$+ E_2B_2 + E_1B_1 + B_3E_3$$
$$+ B_1E_1 + E_2B_2 + E_3B_3$$

$$= 4(\vec{B}\cdot\vec{E})$$

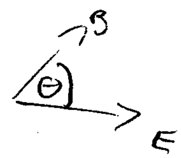
$\Rightarrow$ $\vec{B}\cdot\vec{E}$ is Lorentz invariant

c) **Bonus Question [10 pts]**: Show that $S^2 - c^2\varepsilon^2$ is also Lorentz invariant, where S is the magnitude of the Poynting vector and ε is the energy density of the fields.

$$S^2 - c^2\varepsilon^2$$

$$= \frac{c^2}{64\pi^2}\left[4|\vec{E}\times\vec{B}|^2 - \left(|\vec{E}|^2 + |\vec{B}|^2\right)^2\right]$$

$$|\vec{E}\times\vec{B}|^2 = |\vec{E}|^2|\vec{B}|^2\sin^2\theta$$



$$= |\vec{E}|^2|\vec{B}|^2(1-\cos^2\theta)$$

$$= |\vec{E}|^2|\vec{B}|^2 - (\vec{E}\cdot\vec{B})^2$$

$$= \frac{c^2}{64\pi^2}\left[\underbrace{4|\vec{E}|^2|\vec{B}|^2} - 4(\vec{E}\cdot\vec{B})^2 - \underbrace{\left(|\vec{E}|^2+|\vec{B}|^2\right)^2}\right]$$

$$-\left(|\vec{E}|^2 - |\vec{E}|\right)^2$$

$$= \frac{c^2}{64\pi^2}\left[-4\underbrace{(\vec{E}\cdot\vec{B})^2}_{\text{Lorentz Inv.}} - \underbrace{\left(|\vec{E}|^2 - |\vec{B}|^2\right)^2}_{\text{Lorentz Inv}}\right] \Rightarrow \text{Lorentz inv.!}$$