PHYS 327 PRELIM 2

Prof. Itai Cohen, Fall 2009

Monday, 11/23/09

Name: SOLUTIONS

Read all of the following information before starting the exam:

- Put your name on the exam now.
- Show all work, clearly and in order, if you want to get full credit.
- Circle or otherwise indicate your final answers.
- The first problem is more conceptual, the next two are more computational.
- Question 3 (c) is a bonus question worth 10 points. The total exam score cannot exceed 100, but the bonus question can help you make up points lost elsewhere.

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- It is your responsibility to make sure that you have all of the pages!
- Good luck!
Question 1: Rectangular Waveguide [50 Points]

Consider the propagation of TE waves in a rectangular waveguide with $a > b$:

All field components have the $z$ and $t$ dependence $e^{i(k_x x - \omega t)}$ and can be described in terms of $B_z^0$, which satisfies Helmholtz’ equation

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2 \right) B_z^0 = 0. \quad (1)$$

Together with the boundary conditions $\partial B_z^0/\partial n|_{z = 0} = 0$, this gives the following solution:

$$B_z^0 = B^0 \cos \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right), \quad (2)$$

where $m, n$ are positive integers, at least one of which must be nonzero.

a) [10 pts] Write down an expression for the cut-off frequency $\omega_c \equiv \omega_{mn}$

$$- \left( \frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right) B_z^0 = k_c^2 B_z^0$$

$$\Rightarrow - \left( \frac{m^2 \pi^2}{a^2} - \frac{n^2 \pi^2}{b^2} \right) = k_c^2 \Rightarrow k_c^2 = \frac{1}{n^2} \left( \frac{n^2}{a^2} + \frac{n^2}{b^2} \right)$$

$$b_c^2 = \frac{\omega_c^2}{c^2} \Rightarrow \omega_c = \frac{1}{n^2} \sqrt{\left( \frac{n\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2}$$
b) [10 pts] Consider the mode TE$_{34}$.

(i) Calculate the corresponding cutoff frequency $\omega_{mn}$.

$$\omega_{34} = \pi c \sqrt{\frac{\omega^2}{\alpha^2} + \frac{16}{b^2}}$$

For each of the following cases, does the mode propagate or not? If not, what is the physical reason?

(ii) $\omega > \omega_c$

(iii) $\omega < \omega_c$

(ii) $\omega > \omega_c$ propagates

(iii) $\omega < \omega_c$ does not propagate.

(Several good explanations)

- We cannot find an angle of entry that would set up the required wavelengths along the $x$ and $y$ direction.
- The group velocity becomes imaginary.
- $\omega < \omega_c, \ h_0 < h_c \Rightarrow v_g = \sqrt{h_0^2 - h_c^2}$ becomes imaginary.
c) [20 pts] How should I pick \( \frac{a}{b} \) to maximize the range of frequencies over which only \( \text{TE}_{10} \) propagates? What is this maximum range? Make sure you explain your reasoning.

(This is called "maximizing the single-mode bandwidth" and is desirable for many applications.)

\[
\text{Let } r = \frac{a}{b} \Rightarrow \omega_{mn} = \frac{\pi c}{a} \sqrt{m^2 + n^2 r^2}
\]

As we increase \( \omega \), after \( \text{TE}_{10} \), the next mode to become available for propagation is either \( \text{TE}_{20} \) or \( \text{TE}_{11} \):

\[
\omega_{10} = \frac{\pi c}{a}, \quad \omega_{20} = \frac{\pi c}{a} \times 2, \quad \omega_{11} = \frac{\pi c}{a} \sqrt{1 + r^2}
\]

\[
\frac{\omega}{\pi c/a}
\]

\[
\frac{2}{\pi c/a}
\]

\[
\frac{1}{\pi c/a}
\]

\[
0 \quad \sqrt{3}
\]

\[
\Rightarrow \text{Here for } \frac{a}{b} > \sqrt{3}, \text{ only } \text{TE} \text{ will propagate for max range.}
\]

\[
\frac{\pi c}{a} < \omega < 2 \frac{\pi c}{a}
\]

(Picking \( \frac{a}{b} \approx 2 \) is also acceptable answer)
d) [10 pts] Using \( v_g = \frac{\partial \omega}{\partial k_g} \), show that the group velocity of a TE\(_{mn}\) mode is

\[
v_g = \frac{c}{\omega} \sqrt{\omega^2 - \omega_{mn}^2} < c. \tag{3}
\]

\[
\frac{\omega^2}{c^2} = h_0^2 = h_g^2 + h_c^2
\]

\[
\omega = c \sqrt{\frac{k_g^2 + k_c^2}{k_g^2 + k_c^2}}
\]

\[
\frac{\omega}{k_g} = \frac{c k_g}{\sqrt{k_g^2 + k_c^2}} = \frac{c \sqrt{h_0^2 - h_c^2}}{\omega/c}
\]

\[
= \frac{c^2}{\omega} \left( \frac{1}{c} \sqrt{\omega^2 - \omega_{mn}^2} \right)
\]

\[
v_g = \frac{c}{\omega} \sqrt{\omega^2 - \omega_{mn}^2}
\]
Question 2: Energy Traveling in Rectangular Waveguide [30 Points]

Consider a TE$_{10}$ mode traveling down the rectangular waveguide of Question 1, with $B_z^0$ given by eqn. (2).

_Hints:_

_Throughout this question, you may find the following useful:_

\[
\int_0^a \sin^2 \left( \frac{m\pi x}{a} \right) \, dx = \int_0^a \cos^2 \left( \frac{m\pi x}{a} \right) = \frac{a}{2}
\]

_You may also want to make use of the time-average product theorem:_

\[
\langle F \cdot G \rangle \rightarrow \frac{1}{2} F_0 \cdot G_0^* = \frac{1}{2} F_0^* \cdot G_0,
\]

_where $F_0, G_0$ are complex amplitudes and the multiplication could be any kind, including vector cross- and dot-product._

— Questions on the next pages. —
a) [5 pts] Recall that we derived in lecture how the longitudinal field components in a waveguide determine all other components:

\[ E^0_z = \frac{i}{k_c^2} \left( k_0 \frac{\partial B^0_z}{\partial y} + k_g \frac{\partial E^0_x}{\partial x} \right) \]

\[ E^0_y = -\frac{i}{k_c^2} \left( k_0 \frac{\partial B^0_y}{\partial x} - k_g \frac{\partial E^0_z}{\partial y} \right) \]

\[ B^0_x = -\frac{i}{k_c^2} \left( k_0 \frac{\partial E^0_z}{\partial x} - k_g \frac{\partial B^0_z}{\partial y} \right) \]

\[ B^0_y = \frac{i}{k_c^2} \left( k_0 \frac{\partial E^0_z}{\partial x} + k_g \frac{\partial B^0_z}{\partial y} \right) \]

Use these relations to find all field components of the TE\(_{10}\) mode.

\[ E^0_z = 0 \quad \Rightarrow \quad E^0_x = 0 \quad \Rightarrow \quad E^0_y = \frac{i \alpha^2}{\pi^2} k_0 B^0_s \sin \left( \frac{\pi x}{a} \right) \frac{\pi}{\alpha} \]

\[ B^0_x = -\frac{i \alpha^2}{\pi^2} \frac{(-1)}{\pi} k_g B^0(-1) \sin \frac{\pi y}{a} \frac{\pi}{\alpha} = -\frac{i \alpha}{\pi} k_g B^0 \sin \frac{\pi x}{a} \]

\[ B^0_y = 0 \]
b) [5 pts] Show that
\[ \langle S \rangle_{10} = e_2 \frac{c}{8\pi} \left( \frac{a}{\pi} B_0 \right)^2 k_0 k_g \sin^2 \left( \frac{\pi x}{a} \right) \]

\[ \vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{H} \quad \rightarrow \quad \langle \vec{S} \rangle = \frac{c}{8\pi} \Re \left( \vec{E}_0 \times \vec{B}_0^* \right) \]

\[ \langle \vec{S} \rangle = \frac{c}{8\pi} \Re \left( \begin{pmatrix} 0 \\ \frac{i a}{\pi} \cdot k_0 \cdot B_0 \sin \frac{\pi x}{a} \\ 0 \end{pmatrix} \times \begin{pmatrix} \frac{i a}{\pi} \cdot k_0 \cdot B_0 \sin \frac{\pi x}{a} \\ 0 \\ B_0 \cos \frac{\pi x}{a} \end{pmatrix} \right) \]

\[ \frac{c}{8\pi} \frac{a^2}{\pi^2} \left( \frac{B_0}{2} \right)^2 \sin^2 \left( \frac{\pi x}{a} \right) k_0 k_g \varepsilon_z \]

\[ \langle \hat{S}_{10} \rangle = k_0 k_g \left( \frac{c a^2}{8\pi^3} \left( \frac{B_0}{2} \right)^2 \sin^2 \left( \frac{\pi x}{a} \right) \right) \]

c) [5 pts] Calculate the total power \( P_{10} \) transmitted by the mode.

\[ |\vec{S}| = \text{Power}_\text{area} \quad \Rightarrow \quad \text{Power transmitted in } \varepsilon \text{ direction} = \int (\text{cross-section}) \cdot |\vec{S}| \]

\[ P_{10} = \int_0^a b \, dx \langle \vec{S}_{10} \rangle = \frac{c}{16\pi} k_0 k_g a^3 b \left( \frac{B_0}{2} \right)^2 \]
d) [5 pts] Show that the time-averaged energy-density of the electromagnetic fields of the TE$_{10}$ mode is

$$\langle \varepsilon \rangle_{10} = \frac{(B_0^0)^2}{16 \pi^2} \left\{ (k_0^2 + k_g^2) \alpha^2 \sin^2 \left( \frac{\pi x}{a} \right) + \cos^2 \left( \frac{\pi x}{a} \right) \right\}$$

$$\varepsilon = \frac{1}{8 \pi} \left( \vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B} \right) \equiv \triangle \quad \langle \varepsilon \rangle = \frac{1}{16 \pi} \left( \left| \vec{E}^0 \right|^2 + \left| \vec{B}^0 \right|^2 \right)$$

$$\langle \varepsilon \rangle = \frac{1}{16 \pi} \left( \sum \varepsilon_{1i} \varepsilon_{1i} + \sum B_i B_i \right)$$

$$\langle \varepsilon \rangle = \frac{1}{16 \pi} \left( \left( \frac{(B_0^0)^2}{16 \pi^2} \left( \frac{k_0^2}{\pi^2} \sin^2 \left( \frac{\pi x}{a} \right) + \cos^2 \left( \frac{\pi x}{a} \right) \right) + \frac{k_g^2}{\pi^2} \sin^2 \left( \frac{\pi x}{a} \right) \right) \right)$$

$$\langle \varepsilon \rangle = \frac{(B_0^0)^2}{16 \pi^2} \left( \left( \frac{k_0^2}{\pi^2} \sin^2 \left( \frac{\pi x}{a} \right) + \cos^2 \left( \frac{\pi x}{a} \right) \right) \right)$$

$$\langle \varepsilon \rangle = \frac{(B_0^0)^2}{16 \pi^2} \left( \frac{k_0^2}{\pi^2} + \frac{k_g^2}{\pi^2} \sin^2 \left( \frac{\pi x}{a} \right) + \cos^2 \left( \frac{\pi x}{a} \right) \right)$$

$$\sum_{\text{cross sectional}} \alpha \beta = \frac{(B_0^0)^2 \alpha \beta}{16 \pi^2} \left[ \frac{k_0^2}{\pi^2} + \frac{1}{\pi^2} \right]$$

$$= \frac{(B_0^0)^2 \alpha \beta}{16 \pi^3} \frac{k_0^2}{\pi^2}$$

e) [5 pts] Calculate the time-averaged energy-density per unit length along the waveguide of the mode.

$$\overbrace{\sum}^{\text{cross sectional}} \frac{\alpha \beta}{\pi} \frac{(B_0^0)^2}{16 \pi^2} \left[ \frac{k_0^2}{\pi^2} + \frac{1}{\pi^2} \right]$$
f) [5 pts] Using your answers to parts (c) and (e), show that the energy in the TE_{10} mode travels at the group velocity (defined in part (d) of Question 1).

\[
\text{Power} = \frac{\text{energy}}{\text{time}} \quad \int <\varepsilon> \, da = \frac{\text{energy}}{\text{energy}}
\]

\[
2 \frac{\text{Power}}{\int <\varepsilon> \, da} = \frac{\text{length}}{\text{time}} = \text{speed at which energy travels!}
\]

\[
\nu = \frac{c}{16 \pi^3} \frac{b_0}{k_0} k_0 \alpha^3 b \left(8^o\right) - \frac{16 \pi^3}{(8^o)^2 b_0 \alpha^3 k_0^2}
\]

\[
= \frac{c k_0}{k_0}
\]

\[
\nu_g = \frac{1}{k_0} \sqrt{\frac{c k_0^2 - c k_c^2}{}} = \frac{c k_0}{k_0}
\]

\[
\boxed{\nu = \nu_g}
\]
Scalar Invariants of Electromagnetic Fields [20 Points]

a) [10 pts] Recall that the dual field tensor $G_{\mu\nu}$ can be obtained from the electromagnetic field tensor $F_{\mu\nu}$ by replacing $E \rightarrow B$ and $B \rightarrow -E$. Write down $F_{\mu\nu}$ and $G_{\mu\nu}$.

$$F_{\mu\nu} = \begin{bmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{bmatrix}$$

$$G_{\mu\nu} = \begin{bmatrix} 0 & -E_3 & E_1 & -iB_1 \\ -E_3 & 0 & -E_1 & -iB_2 \\ E_1 & -E_1 & 0 & -iB_3 \\ iB_1 & iB_2 & iB_3 & 0 \end{bmatrix}$$
b) [10 pts] Show that $\mathbf{E} \cdot \mathbf{B}$ and $E^2 - B^2$ are invariant under Lorentz transformations.

We will use the Lorentz invariants $Tr(F^2)$ and $Tr(G^2)$.

\[ Tr(F^2) = F_{ij}F_{ij} = -\sum_{i,j}(F_{ii}^2) \]

\[ = B_1^2 + B_2^2 + B_3^2 - E_1^2 - E_2^2 - E_3^2 = -(E^2 - B^2) \]

\[ \Rightarrow (E^2 - B^2) \text{ is a Lorentz invariant} \]
$$\text{Tr} \left( \hat{A} \hat{P} \right) = B_2 E_3 + B_2 E_2 + E_1 B_1$$
$$+ E_3 B_3 + E_1 B_1 + E_2 B_2$$
$$+ E_2 B_2 + E_1 B_1 + B_3 E_3$$
$$+ B_1 E_1 + E_2 B_2 + E_3 B_3$$

$$= Y \left( \vec{E} \cdot \vec{E} \right)$$

$$\implies \vec{B} \cdot \vec{E} \text{ is Lorentz invariant}$$
c) **Bonus Question [10 pts]**: Show that \( S^2 - c^2 \varepsilon^2 \) is also Lorentz invariant, where \( S \) is the magnitude of the Poynting vector and \( \varepsilon \) is the energy density of the fields.

\[
S^2 - c^2 \varepsilon^2 = \frac{c^2}{64 \pi^2} \left[ 4 |\vec{E} \times \vec{B}|^2 - (|\vec{E}|^2 + |\vec{B}|^2)^2 \right]
\]

\[
|\vec{E} \times \vec{B}|^2 = |\vec{E}|^2 |\vec{B}|^2 \sin^2 \theta
\]

\[
= |\vec{E}|^2 |\vec{B}|^2 (1 - \sin^2 \theta)
\]

\[
= |\vec{E}|^2 |\vec{B}|^2 - (\vec{E} \cdot \vec{B})^2
\]

\[
= \frac{c^2}{64 \pi^2} \left[ 4 |\vec{E}|^2 |\vec{B}|^2 - 4 (\vec{E} \cdot \vec{B})^2 - (|\vec{E}|^2 + |\vec{B}|^2)^2 \right]
\]

\[
- (|\vec{E}|^2 - |\vec{E}|^2)^2
\]

\[
= \frac{c^2}{64 \pi^2} \left[ - 4 (\vec{E} \cdot \vec{B})^2 - (|\vec{E}|^2 - |\vec{E}|^2)^2 \right] \Rightarrow \text{Lorentz Inv.}
\]