Read all of the following information before starting the exam:

- Put your name on the exam now.
- Show all work, clearly and in order, if you want to get full credit.
- It is generally a good idea to read the entire exam through before starting!
- Box or otherwise indicate your final answers.
- The maximum score you can obtain even with extra credit is 100.

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- It is your responsibility to make sure that you have all of the pages!
- Good luck!
Question 1: Dipole scattering (15 pts)
The color of Earth’s sky results from sunlight scattering off air molecules. Since most air molecules (oxygen and nitrogen) are diatomic, to a good approximation we can treat them as little dipole antennas: incident sunlight of frequency $\omega$ causes charges in the molecules to oscillate with the same frequency, and the dominant form of radiation produced is dipole radiation. As discussed in class, the Poynting vector for the radiation produced by a dipole moment is

$$ S = \dot{p}^2 \frac{\sin^2 \theta}{4\pi c^3 R^2} e_r \quad (1) $$

where $\dot{p}$ denotes the second time derivative of the dipole moment of the oscillating charges, $e_r$ denotes the radial unit vector, $\theta$ is defined relative to the dipole axis, and $R$ indicates the distance from the dipole.

a) (5pts) For a sinusoidally oscillating dipole with frequency $\omega$, use the above result to relate $S$ to the frequency $\omega$. Explain what the implications are for scattering of blue versus red light.

For $p \propto e^{i \omega t}$, then $\dot{p} \propto \omega^2 p \propto \omega^2 \Rightarrow S \propto \omega^4$

So higher frequencies have larger radiation intensity, so more strongly scattered. Blue light is more scattered more than red light.

b) (5pts) On the axes shown, sketch a polar plot of the radiation intensity (i.e. the Poynting vector magnitude) produced by this dipole, for fixed $R$, $\omega$.
c) (3pts) As shown in the picture below light is coming from your left and is exciting the molecules in the atmosphere directly in front you. Along which of the three directions (x,y,z) are the charges in the molecules accelerated?

\[ x, z \]

\[ \text{Diagram showing light coming from the left.} \]

d) (2pts) Which orientation of dipoles delivers the greatest amount of radiation energy to your eye?

\[ x \text{ direction} \]

\[ z \text{- direction} \]
Question 2: Fruit flies in a Square Waveguide (45pts)

The researchers at Cohen Airborne Insect Industries have decided to construct a square waveguide that is filled with a swarm of fruit flies. To a good approximation this swarm is a non-conducting dielectric fluid, with dielectric constant \( \epsilon > 1 \), as shown in the figure below. The side length of the waveguide is \( a \), and you may assume the boundary conditions are the same as for a vacuum waveguide.

The researchers have asked you to examine the electrodynamics of a TM mode inside this waveguide.

![Diagram of a square waveguide with a dielectric fluid]  

a) (5pts) Write down the boundary conditions for the electric and magnetic fields in terms of the surface normal \( n \). Can TEM modes propagate in this waveguide? Briefly explain your answer.

\[
\begin{align*}
\nabla \times \mathbf{E} &= \mathbf{0}, \\
\mathbf{A} \cdot \mathbf{n} &= 0
\end{align*}
\]

**TEM modes do not propagate, since they cannot satisfy \( \mathbf{A} \cdot \mathbf{n} = 0 \) everywhere.**

b) (2pts) In the presence of the fruit flies the wave equation satisfied by the electric and magnetic fields \( \mathbf{E}(x,t) \) and \( \mathbf{B}(x,t) \) is

\[
\left( \nabla^2 - \frac{n^2}{c^2} \frac{\partial^2}{\partial t^2} \right) \Psi = 0
\]

where \( \Psi = E_{x,y,z} \) or \( B_{x,y,z} \) and \( n^2 = \epsilon \). To examine the TM or TE modes in the waveguide, as usual we adopt the wave equation ansätze

\[
\begin{align*}
\mathbf{E}(x,t) &= E_0(x,y)e^{i(k_gz-\omega t)}, \\
\mathbf{B}(x,t) &= B_0(x,y)e^{i(k_gz-\omega t)}.
\end{align*}
\]

Find the wave equation for \( E_0 \) and \( B_0 \) in terms of the transverse Laplacian \( \nabla^2_t \) and \( k^2_g = n^2 k_0^2 - k_\perp^2 \), where \( k_0 = \omega/c \).

\[
\begin{align*}
\left( \nabla^2_t - k_0^2 + \frac{\omega^2}{\epsilon} \right)(\frac{E_0}{k_0}) &= 0, \\
\epsilon \left( \nabla^2_t + k_\perp^2 \right)(\frac{B_0}{k_0}) &= 0.
\end{align*}
\]
c) (10pts) In the Cartesian coordinates shown, we write $E^0 = (E_t, E_z)$ and $B^0 = (B_t, B_z)$. Just as for the vacuum case, the transverse electric and magnetic fields $E_t$ and $B_t$ can be expressed in terms of either $E_z$ or $B_z$. For a TM mode in particular, one finds

$$E_t = \frac{ik_g}{\kappa_0} \nabla_t E_z, \quad E_t = -\frac{k_g}{n^2k_0} (e_z \times B_t).$$

(4)

Write down the boundary conditions for $E_z$, and then find the general solution to the wave equation for $E_z$, subject to these boundary conditions. In doing so, show that the dispersion relation $k_g(\omega)$ for TM modes is

$$k_g(\omega) = \sqrt{\left(\frac{n\omega}{c}\right)^2 - \left(\frac{\pi}{a}\right)^2 (k^2 + l^2)}, \quad k, l \in \mathbb{Z}^+. \quad (5)$$

Boundary conditions

$$E_z(0, y) = E_z(a, y) = E_z(x, 0) = E_z(x, a) = 0$$

We have wave eqn

$$(\nabla_t^2 + k_c^2) E_z(x, y) = 0$$

General solution is

$$E_z(x, y) = \sum_{k_x, k_y} A_{k_x, k_y} e^{\pm ik_x x} e^{\pm ik_y y}$$

$$k_c^2 = k_x^2 + k_y^2$$

$$= \sum_{k_x, k_y} A_{k_x, k_y} \sin\left(\frac{k_x x}{a}\right) \sin\left(\frac{k_y y}{a}\right), \quad k_x, k_y = 1, 2, \ldots.$$

which satisfies bc's. We require

$$k_c^2 = \left(\frac{\omega}{c}\right)^2 + \left(\frac{k_x x}{a}\right)^2$$

ie

$$k_g(\omega) = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(k_x l + k_y l^2\right)}.$$
d) (10pts) Use your result to write down \( E_z(x, y) \) for the lowest TM mode and its corresponding cut-off frequency \( \omega_c \). Hence find \( E_0(x, y) \) and \( B_z(x, y) \) for this mode. (You may express your answer in terms of \( k_0, k_c, k_y, n, \) or \( a \) as is convenient.)

We require \( \frac{n \omega}{c} > \sqrt{\frac{k_0^2}{n^2} + k_y^2} = \omega_c \) for \( k = l = 1 \) mode to propagate.

Lowest mode is \( k = l = 1 \), with

\[
\omega_c = \sqrt{2} \left( \frac{\pi}{a} \right)
\]

\[
E_z = E_0 \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{a} \right)
\]

From (4),

\[
E_x = \frac{i k_y E_0 \left( \frac{\pi}{a} \right)}{\sqrt{\omega_c^2 - k_y^2}} \left( \begin{array}{c}
\cos \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{a} \right) \\
\sin \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi y}{a} \right)
\end{array} \right)
\]

\( E \times E \)

\[
B_x E_y - B_y E_x = -\frac{\omega_c}{k_y} E_0
\]

\( B_0 = \frac{\left( \frac{\omega}{\omega_c} \right) E_0 \left( \frac{\pi}{a} \right)}{\sqrt{\omega_c^2 - k_y^2}} \left( \begin{array}{c}
\sin \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi y}{a} \right) \\
\cos \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{a} \right)
\end{array} \right)
\)
e) (8pts) Compute the time-averaged Poynting vector for this mode. In which direction does time-averaged power propagate?

\[ \langle S \rangle = \frac{c}{8\pi} E \times B^t \]

\[ = \frac{c}{8\pi} \left( E_0 \left( \frac{2\pi k_B}{\hbar c^2} \cos(\frac{\pi x}{a}) \sin(\frac{\pi y}{a}) \right), \frac{2\pi k_B}{\hbar c^2} \sin(\frac{\pi x}{a}) \cos(\frac{\pi y}{a}), \sin(\frac{\pi x}{a}) \frac{2\pi k_B}{\hbar c^2} \right) \]

\[ \times -\frac{\hbar^2 k}{\hbar^2 c^2} \left( -\sin(\frac{\pi x}{a}) \cos(\frac{\pi y}{a}), \cos(\frac{\pi x}{a}) \sin(\frac{\pi y}{a}), 0 \right) \]

\[ = \frac{c}{8\pi} E_0^2 \frac{\hbar^2 k}{\hbar^2 c^2} \left( \sin^2(\frac{\pi x}{a}) \cos^2(\frac{\pi y}{a}) + \cos^2(\frac{\pi x}{a}) \sin^2(\frac{\pi y}{a}) \right) \]

(real pt. convention)

\[ \langle S \rangle \text{ power propagates in } +z \text{ direction.} \]
f) (5pts) The fruit flies are particularly sensitive to EM radiation of frequency $\omega_0$. Find the constraints on $a$ such that only the lowest TM mode can propagate down the waveguide at frequency $\omega_0$. (Your result should be expressed in the form of inequalities that $a$ must satisfy.)

\[
\begin{align*}
\text{Dispersion relation } & \quad k_0^2 = \sqrt{\left(\frac{\omega_0}{c}\right)^2 - \left(k_0^2 + \frac{1}{(2)^2}\right) \left(\frac{\mu_n}{\varepsilon_n}\right)^2} \\
\text{For only lowest mode to propagate, we require } & \quad \left(\frac{\omega_0}{c}\right)^2 > 2\left(\frac{\mu_n}{\varepsilon_n}\right)^2 \\
& \quad \text{and } \left(\frac{\omega_0}{c}\right)^2 < (2^2 + 1) \left(\frac{\mu_n}{\varepsilon_n}\right)^2 = 5\left(\frac{\mu_n}{\varepsilon_n}\right)^2 \\
\text{Hence } & \quad \sqrt{5 \frac{\mu_n}{\varepsilon_n}} > a > \sqrt{2 \frac{\mu_n}{\varepsilon_n}}
\end{align*}
\]

g) (5pts) Suppose $a$ has been chosen to satisfy the inequalities of part (f), but the fruit flies now bore a small hole through the waveguide and escape! Show that if the fruit flies' refractive index was $n \geq \sqrt{5/2}$, then once they escape no TM modes can propagate at frequency $\omega_0$. (Hint: Use your results for (f) to find a lower bound on $\pi/a$ and apply it to the new dispersion relation.)

\[
\begin{align*}
\text{New dispersion relation } & \quad k_0^2 = \sqrt{\left(\frac{\omega_0}{c}\right)^2 - \left(k_0^2 + \frac{1}{(2)^2}\right) \left(\frac{\mu_n}{\varepsilon_n}\right)^2} \\
\text{and from (f), } & \quad \left(\frac{\omega_0}{c}\right)^2 > \frac{1}{5} \left(\frac{\omega_0}{c}\right)^2 \\
\text{so then } & \quad \left(\frac{\omega_0}{c}\right)^2 - \left(k_0^2 + \frac{1}{(2)^2}\right) \left(\frac{\mu_n}{\varepsilon_n}\right)^2 \leq \left(\frac{\omega_0}{c}\right)^2 \left[1 - \left(k_0^2 + \frac{1}{(2)^2}\right) \left(\frac{\mu_n}{\varepsilon_n}\right)^2\right] \quad k_0^2 = 1, 2, \ldots \\
& \quad \leq 0 \quad \forall \quad k_0^2 = 1, 2, \ldots \\
& \quad \text{if } \quad n \geq \sqrt{\frac{5}{2}}
\end{align*}
\]
Question 3: Ion Engine (40pts)
An experimental new form of deep-space propulsion operates by accelerating a beam of ions up to a speed \( v \) close to the speed of light \( c \). Each ion in the beam travels at the same velocity \( v \). Before the beam reaches the nozzle of the engine, it passes through a thin-walled pipe which is comprised of a material with dielectric and diamagnetic constants \( \epsilon > 1 \) and \( \mu > 1 \) respectively. The pipe has internal radius \( a \) and wall thickness \( \delta \ll a \), as shown in the figure below. The engineers designing the engine have asked you to compute the pressures produced on the pipe by the ion beam’s electromagnetic fields.

\[ J = \rho v. \]  \hspace{1cm} (6)

a) (5pts) Suppose that the charge density of the ion beam, as measured in the rest frame of the pipe, is \( \rho \) so that the current density of the beam in this frame is

We have \( \int D \cdot dA = \int \epsilon_0 \varepsilon \, dV \text{ and by cylindrical symmetry} \)
\[ \varepsilon = \frac{2 \pi a^2 \rho}{r} \]

Similarly \( \int H \cdot dl = \mu_0 \int J \cdot ds \) so \( J \text{ symmetry} \)
\[ H = \frac{2 \pi a^2 \rho}{r} \frac{\xi}{\xi_0} \]

Hence
\[ i) \quad \xi(r) = \frac{2 \pi a \rho}{r} \xi_0; \quad \xi(a) = \frac{2 \pi a \rho}{r} \xi_0 \]
\[ ii) \quad \xi(r) = \frac{2 \pi a \rho}{r} \xi_0; \quad \xi(a) = \frac{2 \pi a \rho}{r} \xi_0 \]
\[ \xi(r) = \frac{2 \pi a \rho}{r} \xi_0; \quad \xi(a) = \frac{2 \pi a \rho}{r} \xi_0 \]
b) (15pts) Using your results from part (b), find the Maxwell stress tensor component $T_{xx}$ on each side of the pipe's interior and exterior surfaces at the point $\theta = 0$ (so $e_r = e_x$, $e_\theta = e_y$). Hence show that the pressure in the $x$ direction on the interior surface is

$$p^\text{in} = -\frac{\sqrt{\pi}}{2} a^2 \rho^2 \left[ 1 - \frac{1}{\varepsilon^1} + \frac{v^2}{c^2} (\mu^2 - 1) \right] , \quad (7)$$

and find the pressure on the exterior surface $p^\text{ex}$. Be sure to indicate the direction in which these pressures act.

$$T_{xx}^\text{in} = \frac{1}{\varepsilon^1} \left[ \varepsilon_k \varepsilon_k \xi_k - \frac{1}{2} (\varepsilon^2 + \varepsilon^1) \right] , \quad \xi_k = \varepsilon_k \varepsilon_k , \quad \varepsilon = \varepsilon_k \varepsilon_k + \varepsilon^2 \quad (\varepsilon^1 - \varepsilon^2)$$

$$= \frac{1}{\varepsilon^1} (\varepsilon^2 - \varepsilon^1)$$

So

$$T_{xx}^\text{wall} = \frac{\sqrt{\pi}}{2} a^2 \rho^2 \left( 1 - \varepsilon^1 \right)$$

$$T_{xx}^\text{wall} = \frac{\sqrt{\pi}}{2} a^2 \rho^2 \left( \frac{1}{\varepsilon^1} - \varepsilon^1 \mu^2 \right)$$

$$T_{xx}^\text{wall} (a + \delta) = \frac{\sqrt{\pi}}{2} a^2 \rho^2 \left( \frac{1}{\varepsilon^1} - \varepsilon^1 \mu^2 \right)$$

$$T_{xx}^\text{wall} (a + \delta) = \frac{\sqrt{\pi}}{2} a^2 \rho^2 \left( 1 - \varepsilon^1 \right)$$

$$T_{xx}^\text{wall} (a + \delta) = \frac{\sqrt{\pi}}{2} a^2 \rho^2 \left[ 1 - \frac{1}{\varepsilon^1} + \mu^2 (\mu^2 - 1) \right]$$

$$T_{xx}^\text{wall} (a + \delta) = \frac{\sqrt{\pi}}{2} a^2 \rho^2 \left[ 1 - \frac{1}{\varepsilon^1} + \mu^2 (\mu^2 - 1) \right]$$

$$p^\text{in} = \varepsilon_k \cdot \left[ T_{xx}^\text{wall} (\varepsilon_k) + T_{xx}^\text{wall} (\varepsilon_k) \right] = -T_{xx}^\text{in} + T_{xx}^\text{wall} \Big|_a$$

$$= -\frac{\sqrt{\pi}}{2} a^2 \rho^2 \left[ 1 - \frac{1}{\varepsilon^1} + \mu^2 (\mu^2 - 1) \right]$$

$$p^\text{ex} = \varepsilon_k \left[ T_{xx}^\text{wall} (\varepsilon_k) + T_{xx}^\text{wall} (\varepsilon_k) \right] = T_{xx}^\text{wall} - T_{xx}^\text{wall} \Big|_{a+\delta}$$

$$= \frac{\sqrt{\pi}}{2} a^2 \rho^2 \left[ 1 - \frac{1}{\varepsilon^1} + \mu^2 (\mu^2 - 1) \right]$$
c) (10pts) Using the fact that the differential area on a cylinder \( dA = r \, d\theta \, dz \), and then expanding your results in the small parameter \( \delta/a \), find the leading order net differential force on the pipe in the \( x \)-direction at angle \( \theta \) from the \( x \)-axis. I.e. find \( dF_x / d\theta \, dz \).

**Differential force**

\[
d \xi = T \cdot dA
\]

So

\[
dF_x = \sum_{x} T \cdot \xi \cdot r \, d\theta \, dz \quad \text{and} \quad \xi = \cos \theta \, \xi_x - \sin \theta \, \xi_y
\]

\[
= \left( p \sin \alpha \cos \theta + p \cos \alpha \sin \theta \right) \cos \theta \, d\theta \, dz
\]

\[
= \frac{\xi}{2} \left( 1 - \frac{1}{\epsilon} + \frac{\alpha^2}{a^2} \right) \cos \theta \, d\theta \, dz
\]

\[
\frac{dF_x}{d\theta \, dz} \sim -\frac{\xi}{2} a^2 \sin \theta \left[ 1 - \frac{1}{\epsilon} + \frac{\alpha^2}{a^2} \right]
\]

\[
\frac{\alpha^4}{\alpha \cos^2} - \frac{\alpha^3}{\alpha \sin^2}
\]

\[
= \frac{\alpha^4 - \alpha^3}{\alpha \cos^2}
\]

\[
= -\frac{\alpha^3 \sin^2}{\alpha \sin^2} \sim -\alpha^2 \sin^2
\]

d) (5pts) What happens to \( dF_x / d\theta \, dz \) as \( \delta \to 0 \). Explain your result.

As \( \delta \to 0 \), \( \frac{dF_x}{d\theta \, dz} \to 0 \). If wall has no thickness, the \( \xi, \theta \) is zero across surface, so there is no pressure.
e) (5pts) The ‘hoop stress’ on the pipe is the global azimuthal compressive or expansive stress produced in the pipe wall due to the local compressive or expansive radial pressures. One may compute the hoop stress by slicing the pipe in the y axis, and then computing the total force acting on the resultant half-pipe in the x-direction (a free body diagram is shown below). The hoop stress is this force divided by the wall area of the half-pipe (to a good approximation when \( \delta \ll a \)). Without finding the hoop stress explicitly, use your result for \( dF_x/d\delta dz \) to argue that the leading order hoop stress is independent of \( \delta \).

![Free body diagram of hoop stress](image)

\[
\text{Hoop stress} \quad \sigma \propto \frac{1}{2\pi L} \int (dF_x/d\delta d\theta) d\delta dz \quad \text{and} \quad dF_x/d\delta d\theta \propto \delta
\]

Hence \( \sigma \propto \frac{\delta}{\delta} \propto \text{const.}! \)
Question 4 (Bonus): "The Boy who Harnessed the Wind" (5 pts)

a) (2pts) What motivates William Kamkwamba to build his windmill? What problems will it solve?

b) (1pt) Who teaches William how to build his windmill?

c) (2pts) What device does William Kamkwamba use to prevent electrical fires in his house?
$\sigma_b = n \cdot P$
$\rho_b = -\nabla \cdot P$
$(D_2 - D_1) \cdot n = 4\pi \rho_f$
$(E_2 - E_1) \times n = 0$

$\frac{K_b}{c} = -n \times M$
$\frac{J_b}{c} = \nabla \times M$
$(B_2 - B_1) \cdot n = 0$
$(H_2 - H_1) \times n = -\frac{4\pi}{c} K_f$

$D = E + 4\pi P$
$P = \chi_e E$
$D = \varepsilon E$

$H = B - 4\pi M$
$M = \chi_m H$
$B = \mu H$

$\Phi(r) = \int_V \frac{\rho(r')}{|r - r'|} dV$

$E = -\nabla \Phi - \frac{1}{c} \partial_t A$

$A(r) = \int_V \frac{J(r')}{|r - r'|} dV$

$B = \nabla \times A$

$Q = \sum_\alpha q_\alpha$
$\Phi^{(1)} = \frac{Q}{r}$
$p = \sum_\alpha q_\alpha r'_\alpha$
$\Phi^{(2)} = \frac{p \cdot r}{r^3}$

$Q_{ij} = \sum_\alpha q_\alpha (3x'_i x'_j \delta_{ij} - r'^2 \delta_{ij})$

$\Phi^{(4)} = \frac{1}{6} \sum_{ij} Q_{ij} \left( \frac{3x_i x_j - r^2 \delta_{ij}}{r^5} \right)$

$\Phi(x, y, z) = \sum_{\alpha, \beta, \gamma} \left| \alpha^2 + \beta^2 + \gamma^2 = 0 \right| e^{\pm \alpha x} e^{\pm \beta y} e^{\pm \gamma z}$

$\Phi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ A_{l} r^l + \frac{B_i}{r^{l+1}} \right] Y_{lm}(\theta, \phi)$

$\Phi(r, \phi, z) = A_0 + B_0 \log r + \sum_{n=1}^{\infty} \left( \left[ A_{r} r^l + \frac{B_i}{r^{l+1}} \right] \cos(n\phi) + \left[ C_{r} r^l + \frac{D_i}{r^{l+1}} \right] \sin(n\phi) \right)$

$+ \sum_{m,n} \left[ A_{mn} J_n(k_m r) + B_{mn} N_n(k_m r) \right] e^{\pm \imath k_m x} e^{\pm \imath x/n}$

$\int_0^L dx \sin \frac{n \pi x}{L} \sin \frac{m \pi x}{L} = \frac{L}{2} \delta_{mn}$

$\int_{-1}^{+1} dx P_l(x) P_m(x) = \frac{2}{2l + 1} \delta_{lm}$

$\int_0^{4\pi} Y_{lm}^*(\theta, \phi) Y_{kn}(\theta, \phi) d\Omega = \delta_{mn} \delta_{lk}$

$\int_0^r r dr J_n(k_m r) J_n(k_l r) = \frac{\rho^2}{2} J_{n+1}^2(k_m \rho) \delta_{ml}$

$F + \frac{d}{dt} \int_V \frac{1}{4\pi c} E \times B = \int_V \nabla \cdot T dV$

$\frac{\partial E}{\partial t} + \nabla \cdot S + E \cdot J = 0$
\[ \langle f(t)g(t) \rangle = \frac{1}{2} f_0 g_0 \]

\[ S = \frac{c}{4\pi} E \times H \]

\[ dF = T \cdot dA \]

\[ g_{\text{field}} = \frac{1}{c^2} S \]

\[ B = ne_k \times E \quad E = -\frac{1}{n} e_k \times B \]

\[ T_{ij} = \frac{1}{4\pi} \left[ E_i E_j + B_i B_j - \frac{1}{2} \delta_{ij}(E^2 + B^2) \right] \quad \nabla \cdot S = \frac{\partial g_{\text{field}}}{\partial t} + \frac{\partial g_{\text{matter}}}{\partial t} \]

\[ Y_0^0 = \sqrt{\frac{1}{4\pi}} \quad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta \quad Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi} \]

\[ Y_2^0 = \sqrt{\frac{5}{16\pi}} (3\cos^2 \theta - 1) \quad Y_2^{\pm 1} = \mp \sqrt{\frac{15}{32\pi}} \sin 2\theta e^{\pm i\phi} \quad Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi} \]

\[ \frac{d}{du} Z_n(u) = \frac{n}{u} Z_n(u) - Z_{n+1}(u) \quad \frac{d}{du} Z_0(u) = -Z_1(u) \]

\[ \left. \frac{\partial E_z^0}{\partial \alpha} \right|_S = 0 \quad E_z^0 \big|_S = 0 \]

\[ \beta = \frac{v}{c} \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}} \quad x_\mu = (x, ic) \quad \partial_\mu = (\nabla, \frac{i}{c} \partial_t) \]

\[ J_\mu = (J, ic\rho) \quad \partial_\mu J_\mu = 0 \quad A_\mu = (A_0, \vec{A}) \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]

\[ \partial_\mu F_{\mu\nu} = \frac{4\pi}{c} J_\nu \quad G_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} \quad \partial_\mu G_{\mu\nu} = 0 \quad F_{\mu\nu} F_{\mu\nu} = 2(B^2 - E^2) \]

\[ G_{\mu\nu} G_{\mu\nu} = 2(E^2 - B^2) \quad F_{\mu\nu} G_{\mu\nu} = 4E \cdot B \]

\[ F = \begin{pmatrix}
0 & B_3 & -B_2 & -iE_1 \\
-B_3 & 0 & B_1 & -iE_2 \\
B_2 & -B_1 & 0 & -iE_3 \\
iE_1 & iE_2 & iE_3 & 0
\end{pmatrix} \quad \lambda(0, 0, \beta) = \begin{pmatrix}
1 \\
\gamma \\
i\beta\gamma \\
\gamma
\end{pmatrix} \]

\[ G = \begin{pmatrix}
0 & -E_3 & E_2 & -iB_1 \\
E_3 & 0 & -E_1 & -iB_2 \\
-E_2 & E_1 & 0 & -iB_3 \\
iB_1 & iB_2 & iB_3 & 0
\end{pmatrix} \quad \lambda(\beta, 0, 0) = \begin{pmatrix}
\gamma & i\beta\gamma \\
1 \\
-i\beta\gamma \\
\gamma
\end{pmatrix} \]