PHYS 3327 PRELIM 2

Prof. Itai Cohen, Fall 2016

Wednesday, 11/21/16 **Name:**

Read all of the following information before starting the exam:

- Put your name on the exam **now**.
- Show all work, clearly and in order, if you want to get full credit.
- Box or otherwise indicate your final answers.
- Question 6 is a extra credit question worth 8 points. The total exam score cannot exceed 100, but the extra credit can help you make up points lost elsewhere.

Problem #	Score
1	/10
2	/10
3	/15
4	/30
5	/35
6 Extra credit	/08
Total	/100

- It is your responsibility to make sure that you have all of the pages!
- Good luck!

Question 1: Transformation of EM fields [10 points]

You measure the electric field $\vec{E}$ and magnetic field $\vec{B}$ at a particular point from an inertial frame $\mathcal{S}$, and find $B = 0$ and $E \neq 0$. Let $\vec{E}'$ and $\vec{B}'$ denote the electric and magnetic fields at the same point as measured from a different inertial frame $\mathcal{S}'$. Circle the correct option(s):

(i) It is possible that $E' = 0$ and $B' \neq 0$.

(ii) It is possible that $E' < B'$.

(iii) It is possible that $B' = 0$ and $E' \neq 0$.

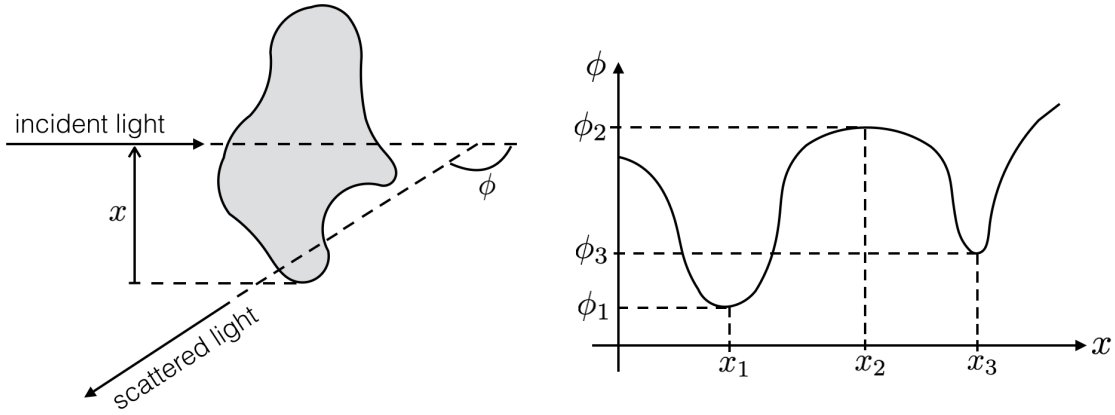
(iv) If both E' and B' are non-zero, then $\vec{E}'$ must be perpendicular to $\vec{B}'$.

(v) It is possible that both E' and B' are zero.

Ans. The key here is to note that the quantities $E^2 - B^2$ and $\vec{E} \cdot \vec{B}$ are frame-invariant. In other words, $E'^2 - B'^2 = E^2 - B^2$ and $\vec{E}' \cdot \vec{B}' = \vec{E} \cdot \vec{B}$. The first equality rules out (i), (ii), and (v). The second equality requires (iv), because the dot product is zero in both frames. Option (iii) is also correct, e.g., if we go to a frame which is moving parallel to the direction of $\vec{E}$. Thus the correct options are (iii) and (iv).

Question 2: Tyndall effect [10 points]

Consider a colloidal suspension in which little rod-shaped particles oriented along z are dispersed in a medium. Figure shows the cross-section of such a particle in the x-y plane. We shine light from the left, which is scattered back by the particles. The scattering angle ϕ is a function of the impact parameter x, as shown in the accompanying plot.



Suppose the reflection- and transmission coefficients involved in the scattering are slowly varying functions of x. Consider the intensity of the scattered light as a function of ϕ.

(a) [5 points] At which value(s) of ϕ do you expect to see intensity peak(s)?

Ans. We expect to see intensity peaks at all extremum points, i.e., at ϕ_1, ϕ_2, and ϕ_3. This is because near any extremum (maximum or minimum), a small variation in ϕ corresponds to a large variation in x. Since the light is uniformly distributed in x, large fractions of the scattered light will be concentrated near the extremum values, producing intensity peaks.

(b) [5 points] If there are multiple peaks, which one should have the maximum intensity and why?

Ans. The maximum intensity should be at $\phi = \phi_2$. This is because it is the broadest extremum (i.e., smallest curvature). The reasoning outlined in (a) then implies that this angle should have the maximum intensity.

Question 3: A moving charge stops [15 points]

A point charge Q traveling along the x axis at a velocity $\vec{v} = 0.6c\hat{x}$ suddenly stops as it reaches $x = 0$ at $t = 0$. It is known that the electric field produced by a charge Q moving at *uniform* velocity v is given by

$$\vec{E}(\vec{r}, t) = \frac{Q}{R^2} \frac{1 - \beta^2}{(1 - \beta^2 \sin^2 \theta)^{3/2}} \hat{n},$$

where $\vec{R} \equiv \vec{r} - \vec{r}_q(t)$, $\vec{r}_q$ being the location of the charge at time t, $\hat{n}$ is a unit vector along $\vec{R}$, and θ is the angle between $\vec{v}$ and $\vec{R}$.

(a) [5 points] Find the magnitude and direction of the electric field at $t = 5$ as measured by someone located at $(x, y, z) = (2c, 3c, 0)$.

Hint: The "news" that the charge has stopped spreads out at the speed of light.

Ans. By $t = 5$ the "news" that the charge has stopped has reached a radius of $5c$ centered at the origin. The point $(2c, 3c, 0)$ is at a distance $\sqrt{(2c)^2 + (3c)^2 + (0)^2} = \sqrt{13}c < 5c$. Therefore, an observer at this location simply sees the field of a stationary charge at the origin at $t = 5$. Hence the electric field he or she would measure is

$$\vec{E} = \frac{Q}{(\sqrt{13}c)^3}(2c\hat{x} + 3c\hat{y}) = \frac{Q}{13c^2}\left(\frac{2}{\sqrt{13}}\hat{x} + \frac{3}{\sqrt{13}}\hat{y}\right).$$

(b) [10 points] Find the magnitude and direction of the electric field at $t = 5$ as measured by someone located at $(x, y, z) = (6c, 4c, 0)$.

Ans. The location $(6c, 4c, 0)$ is at a distance $\sqrt{(6c)^2 + (4c)^2 + (0)^2} = \sqrt{40}c > 5c$. Thus, the observer at this location has no way of knowing that the charge has stopped. Therefore, he/she must measure the electric field of a uniformly moving charge. So the field measured is given by the expression

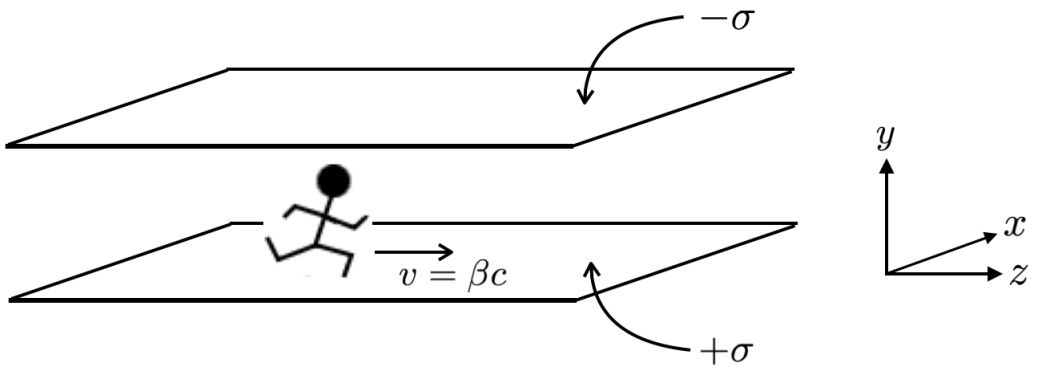
$$\vec{E} = \frac{Q}{R^2} \frac{1 - \beta^2}{(1 - \beta^2 \sin^2\theta)^{3/2}} \hat{n},$$

where $\vec{R}$ connect the points $(6c, 4c, 0)$ from the location *where the charge would be had it continued moving at the velocity* $\vec{v} = 0.6c\hat{x}$*!* That location at $t = 5$ is $(3c, 0, 0)$. Therefore, $\vec{R} = (6c, 4c, 0) - (3c, 0, 0) = (3c, 4c, 0)$, and $\sin\theta = 4/5 = 0.8$. Thus,

$$\vec{E} = \frac{Q}{25c^2} \frac{1 - 0.6^2}{(1 - 0.6^2 \times 0.8^2)^{3/2}} \left(\frac{3}{5}\hat{x} + \frac{4}{5}\hat{y}\right) \approx 0.95 \frac{Q}{25c^2} \left(\frac{3}{5}\hat{x} + \frac{4}{5}\hat{y}\right).$$

Question 4: Running through a capacitor [30 points]

A rich friend of yours owns an infinite parallel plate capacitor, shown below, inside of which he likes to go running every Monday evening. He has asked you to calculate the electric and magnetic fields he will experience inside the capacitor during his run. He cannot run very fast ($\beta \ll 1$, or $\gamma \approx 1$), so that Lorentz contraction and time dilation effects are negligible.



In the capacitor rest frame, the plates have surface charge density $\pm\sigma$, which generate an electric field $\vec{E} = 4\pi\sigma\hat{y}$ between the plates.

For parts (a)–(c) below, you do not need any relativistic transformation of vectors/tensors. Give physical reasoning.

(a) [6 points] What is the surface charge density σ' and the electric field $\vec{E}'$ in your running friend's reference frame, assuming $\gamma \approx 1$?

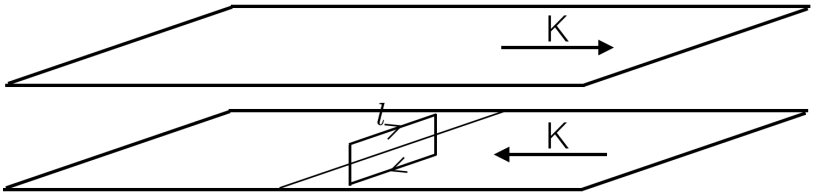
Ans. Since we are neglecting Lorentz contraction and we know that charge is frame-invariant, your friend will "see" the same surface charge density on the capacitor plates, i.e., $\sigma' \approx \sigma$. Hence the electric field is also the same, $\vec{E}' \approx \vec{E} = 4\pi\sigma\hat{y}$.

(b) [6 points] To your friend, each capacitor plate behaves like a current sheet. What is the current *per unit length along* x in each sheet, assuming $\gamma \approx 1$?

Ans. Each plate is moving to the left at speed v, and they have surface charge density σ. Therefore each of them behaves like a current sheet with current per unit length $K = \sigma v$, although the currents are in opposite directions because the plates have opposite charges. The current is in the $+z$ direction for the upper plate, and in the $-z$ direction for the lower plate.

(c) [6 points] Use your result in (b) to calculate the magnetic field $\vec{B}'$ between the plates in your friend's frame. (Give both magnitude and direction!)

Hint: Use integral form of Ampére's law.



Ans. Since the current sheets are infinite, we can convince ourselves by right hand rule, that the magnetic field will be uniform and pointing in the x direction in the space between the plates, and zero outside the plates. Applying Ampere's law to the closed loop shown, we get

$$B'l = \frac{4\pi}{c} Kl = \frac{4\pi}{c} \sigma v l \implies \vec{B}' = 4\pi\sigma\beta\hat{x}.$$

(d) [6 points] Write the EM field tensor $F_{\mu\nu}$ in the capacitor rest frame.

Ans. Using the definition of the field tensor in terms of the electric and magnetic fields, we get

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i4\pi\sigma \\ 0 & 0 & 0 & 0 \\ 0 & i4\pi\sigma & 0 & 0 \end{pmatrix}.$$

(e) [6 points] Compute the field tensor $F'_{\mu\nu}$ in your friend's frame by applying Lorentz transformations on $F_{\mu\nu}$, assuming $\gamma \approx 1$. Identify the electric and magnetic fields $\vec{E}'$ and $\vec{B}'$ from $F'_{\mu\nu}$. Do they agree with your expressions in (a) and (c)?

Ans. The field tensor transforms as $F' = \lambda F \lambda^T$ where λ is the Lorentz transformation matrix. Thus, with $\gamma \approx 1$

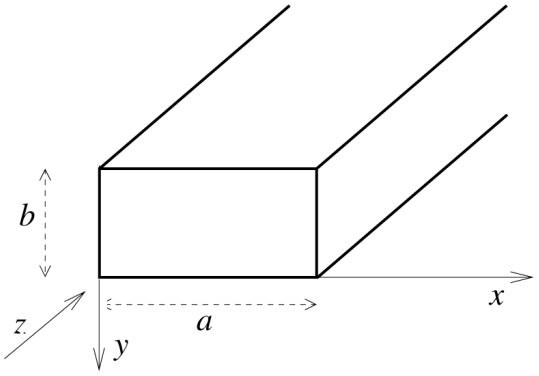
$$F' \approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & i\beta \\ 0 & 0 & -i\beta & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i4\pi\sigma \\ 0 & 0 & 0 & 0 \\ 0 & i4\pi\sigma & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -i\beta \\ 0 & 0 & i\beta & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 4\pi\sigma\beta & -i4\pi\sigma \\ 0 & -4\pi\sigma\beta & 0 & 0 \\ 0 & i4\pi\sigma & 0 & 0 \end{pmatrix}$$

We can identify from F' that $\vec{E}' = -4\pi\sigma\hat{y}$, and $\vec{B}' = 4\pi\sigma\beta\hat{x}$., which indeed agree with the expressions we found in (a) and (c) from physical reasoning.

Question 5: TE modes in a waveguide [35 points]

Consider the propagation of TE waves in a rectangular waveguide of sides a and b.



All field components have the z and t dependence $e^{i(k_g z-\omega t)}$, and can be described in terms of the amplitude of the z-component of the magnetic field B_z^0, which satisfies the equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\omega^2}{c^2}-k_g^2\right)B_z^0=0. \tag{1}$$

Together with the boundary conditions $\partial B_z^0/\partial n|_S=0$, this gives the following solution:

$$B_z^0=B^0\cos\left(m\pi x/a\right)\cos\left(n\pi y/b\right), \tag{2}$$

where m, n are non-negative integers, at least one of which is non-zero.

(a) [10 points] Use Eqs. (1) and (2) to show that the cut-off frequency for the (m,n) mode is given by

$$\omega_{mn}=(c\pi/a)\sqrt{m^2+n^2\left(a/b\right)^2}.$$

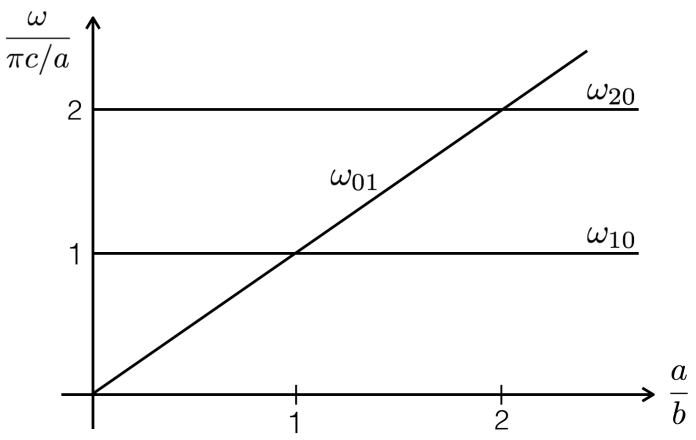
Does the mode propagate for $\omega>\omega_{mn}$ or $\omega<\omega_{mn}$? Explain.

Ans. Substituting the expression in (2) into (1), we get

$$\begin{aligned} &-(m\pi/a)^2-(n\pi/b)^2+(\omega/c)^2-k_g^2=0,\\ \text{or,}\quad &k_g=\sqrt{(\omega/c)^2-(m\pi/a)^2-(n\pi/b)^2}. \end{aligned}$$

For the mode to propagate, k_g must be real, or, $\omega>\omega_{mn}=(c\pi/a)\sqrt{m^2+n^2\left(a/b\right)^2}$.

(b) [10 points] For a given value of a, plot the variation of the cut-off frequencies of the lowest three modes (ω_{01}, ω_{10}, and ω_{20}) with the aspect ratio a/b.



(c) [5 points] For $a/b = 1.5$, find the range of ω over which only the TE_{10} mode propagates.

Hint: Use your plot in (b).

Ans. The (m, n) mode propagates if $\omega > \omega_{mn}$. Thus, only the (1,0) mode will propagate from $\omega > \omega_{10}$ and $\omega < \omega_{mn}$ for all other (m, n). This is true at $a/b = 1.5$ for $c\pi/a < \omega < 1.5\,\pi/a$.

(d) [10 points] How should one pick a/b to maximize the range of ω over which only the TE_{10} mode propagates? What is that maximum range?
(This is called "maximizing the single-mode bandwidth" which is desirable for many applications.)

Ans. From the plot in (b), we see that the range of ω for which only the (1,0) mode propagates increases from 0 at $a/b = 1$ to $c\pi/a$ at $a/b = 2$, then stays constant. Therefore, to maximize the range, we need to choose $a/b \geq 2$, and the maximum range is $c\pi/a < \omega < 2c\pi/a$.

Bonus: "The Boy who Harnessed the Wind" [8 points]

I attest that I read the entire book titled "The boy who harnessed the wind."

Signature _____

(a) [2 points] What motivates William Kamkwamba to build his windmill? What problems will it solve?

Ans. Without light he cannot study at night. Also people need electricity to charge their cell phone batteries. Finally he needs to power the water pump for the well.

(b) [3 points] Who teaches William how to build his windmill?

Ans. He learns it from a book in a library.

(c) [3 points] What device does William Kamkwamba use to prevent electrical fires in his house?

Ans. He builds a simple fuse.

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Formulae

$$\sigma_b = n \cdot P \qquad \rho_b = -\nabla \cdot P \qquad (D_2 - D_1) \cdot n = 4\pi\rho_f \qquad (E_2 - E_1) \times n = 0$$

$$\frac{K_b}{c} = -n \times M \qquad \frac{J_b}{c} = \nabla \times M \qquad (B_2 - B_1) \cdot n = 0 \qquad (H_2 - H_1) \times n = -\frac{4\pi}{c} K_f$$

$$D = E + 4\pi P \qquad P = \chi_e E \qquad D = \varepsilon E$$

$$H = B - 4\pi M \qquad M = \chi_m H \qquad B = \mu H$$

$$\Phi(r) = \int_V dv' \frac{\rho(r')}{|r - r'|} \qquad E = -\nabla\Phi \qquad A(r) = \int_V dv' \frac{J(r')}{|r - r'|} \qquad B = \nabla \times A$$

$$Q = \sum_\alpha q_\alpha \qquad \Phi^{(1)} = \frac{Q}{r} \qquad p = \sum_\alpha q_\alpha r'_\alpha \qquad \Phi^{(2)} = \frac{p \cdot r}{r^3}$$

$$Q_{ij} = \sum_\alpha q_\alpha (3x'_{i\alpha}x'_{j\alpha} - r'^2_\alpha \delta_{ij}) \qquad \Phi^{(4)} = \frac{1}{6}\sum_{ij} Q_{ij}\left(\frac{3x_i x_j - r^2\delta_{ij}}{r^5}\right)$$

$$\Phi(x,y,z) = \sum_{\alpha,\beta,\gamma}\Big|_{\alpha^2+\beta^2+\gamma^2=0} e^{\pm\alpha x} e^{\pm\beta y} e^{\pm\gamma z} \qquad \Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + \frac{B_l}{r^{l+1}}\right] P_l(\cos\theta)$$

$$\Phi(r,\phi,\theta) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_l^m r^l + \frac{B_l^m}{r^{l+1}}\right] Y_l^m(\theta,\phi)$$

$$\begin{aligned}\Phi(r,\phi,z) = {} & A_0 + B_0 \log r + \sum_{n=1}^{\infty}\left(\left[A_n r^n + \frac{B_n}{r^n}\right]\cos(n\phi) + \left[C_n r^n + \frac{D_n}{r^n}\right]\sin(n\phi)\right) \\ & + \sum_{m,n}\Big[A_{mn} J_n(k_m r) + B_{mn} N_n(k_m r)\Big] e^{\pm in\phi} e^{\pm k_m z}\end{aligned}$$

$$\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{\partial A_\phi}{\partial\phi}\right]$$

$$\int_0^L dx \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = \frac{L}{2}\delta_{mn} \qquad \int_0^\pi d\theta \sin\theta P_l(\cos\theta) P_m(\cos\theta) = \frac{2}{2l+1}\delta_{lm}$$

$$\int_{4\pi} Y_l^m(\theta,\varphi) Y_k^{n\,*}(\theta,\varphi) d\Omega = \delta_{mn}\delta_{lk} \qquad \int_0^\rho r dr J_n(k_m r) J_n(k_l r) = \frac{\rho^2}{2} J_{n+1}^2(k_m\rho)\delta_{ml}$$

$$T_{ij} = \frac{1}{4\pi}\left(E_i E_j + B_i B_j - \frac{1}{2}(E^2 + B^2)\delta_{ij}\right) \qquad \mathcal{E} = \frac{1}{8\pi}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}) \qquad \vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{H}$$

$$\vec{\nabla}\cdot\vec{E}=4\pi\rho \qquad \vec{\nabla}\cdot\vec{B}=0 \qquad \vec{\nabla}\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B}=\frac{1}{c}\frac{\partial\vec{E}}{\partial t}+\frac{4\pi}{c}\vec{J}$$

$$\oint\vec{E}\cdot d\vec{a}=4\pi Q_{\text{encl}} \qquad \oint\vec{B}\cdot d\vec{a}=0 \qquad \oint\vec{E}\cdot d\vec{l}=-\frac{1}{c}\frac{d\Phi_B}{dt} \qquad \oint\vec{B}\cdot d\vec{l}=\frac{1}{c}\frac{d\Phi_E}{dt}+\frac{4\pi}{c}I_{\text{link}}$$

$$\Phi(\vec{r},t)=\int\frac{\rho\left(\vec{r}',t-|\vec{r}-\vec{r}'|/c\right)}{|\vec{r}-\vec{r}'|}dv' \qquad \vec{A}(\vec{r},t)=\int\frac{\vec{J}\left(\vec{r}',t-|\vec{r}-\vec{r}'|/c\right)}{c|\vec{r}-\vec{r}'|}dv'$$

$$\Phi=\frac{q}{R-\vec{\beta}.\vec{R}} \qquad \vec{A}=\frac{q\vec{\beta}}{R-\vec{\beta}.\vec{R}} \qquad \vec{E}=q\left[\frac{(\hat{n}-\vec{\beta})(1-\beta^2)}{R^2(1-\vec{\beta}.\hat{n})^3}+\frac{\hat{n}\times((\hat{n}-\vec{\beta})\times\vec{a})}{c^2R(1-\vec{\beta}.\hat{n})^3}\right] \qquad \vec{B}=\hat{n}\times\vec{E}$$

$$\vec{E}=\frac{q}{R_p^2}\frac{1-\beta^2}{(1-\beta^2\sin^2\theta_p)^{3/2}}\hat{n}_p \qquad \vec{B}=\vec{\beta}\times\vec{E}$$

$$E_{r\parallel}^0=\frac{n_2\cos\theta_1-n_1\cos\theta_2}{n_2\cos\theta_1+n_1\cos\theta_2}E_{i\parallel}^0 \qquad E_{t\parallel}^0=\frac{2n_1\cos\theta_1}{n_2\cos\theta_1+n_1\cos\theta_2}E_{i\parallel}^0$$

$$E_{r\perp}^0=\frac{n_1\cos\theta_1-n_2\cos\theta_2}{n_1\cos\theta_1+n_2\cos\theta_2}E_{i\perp}^0 \qquad E_{t\perp}^0=\frac{2n_1\cos\theta_1}{n_1\cos\theta_1+n_2\cos\theta_2}E_{i\perp}^0$$

$$\lambda=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\gamma&i\gamma\beta\\0&0&-i\gamma\beta&\gamma\end{pmatrix} \qquad F=\begin{pmatrix}0&B_3&-B_2&-iE_1\\-B_3&0&B_1&-iE_2\\B_2&-B_1&0&-iE_3\\iE_1&iE_2&iE_3&0\end{pmatrix}$$

$$\vec{E}'_\perp=\gamma(\vec{E}+\vec{\beta}\times\vec{B})_\perp \qquad \vec{E}'_\parallel=\vec{E}_\parallel \qquad \vec{B}'_\perp=\gamma(\vec{B}-\vec{\beta}\times\vec{E})_\perp \qquad \vec{B}'_\parallel=\vec{B}_\parallel$$

$$\vec{E}'\cdot\vec{B}'=\vec{E}\cdot\vec{B} \qquad E'^2-B'^2=E^2-B^2$$

$$\vec{F}=q\left(\vec{E}+\frac{1}{c}\vec{v}\times\vec{B}\right)$$