Solutions to Problem Set 1

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1.3 Birthday Problem

> Suppose there are N people in a room. What is the probability that at least two of them share the same birthday - the same day of the same month?

It's easiest to begin by calculating the probability $p(N)$ that N people in a room all have different birthdays. The probability of at least two people having the same birthday is then $1 - p(N)$. Assuming that birthdays are evenly distributed throughout the year, the probability that, in some arbitrary order, the second person in the room does not have the same birthday as the first is given by 364/365. The probability that a third person does not share a birthday with the first two is then 363/365. Continuing in this vein, we find

$$p(N) = \frac{364}{365} \times \frac{363}{365} \times \ldots \times \frac{365 - N + 1}{365} = \frac{365!}{365^N (365 - N)!}$$

1.4 Russian Roulette

> **Reif §1.5:** In the game of Russian roulette, one inserts a single cartridge into the drum of a revolver, leaving the other five chambers of the drum empty. One then spins the drum, aims at one's head and pulls the trigger.
>
> **(a)** What is the probability of still being alive after playing the game N times?
>
> **(b)** What is the probability of surviving $(N - 1)$ turns in this game and then being shot the N^{th} time on pulls the trigger?
>
> **(c)** What is the mean number of times a player gets the opportunity of pulling the trigger in this macabre game?

(a) The probability of surviving one round of the game is 5/6. Assuming that the chamber is re-spun after each round, the probability of surviving N rounds is then $(5/6)^N$.

(b) The probability of dying immediately after having survived some number of turns is 1/6, so the probability of dying on the N^{th} turn is $(1/6)(5/6)^{N-1}$.

(c) To find the mean number of turns one can play before death, we compute the sum $\sum_n np(n)$, with $p(n)$ the probability of the game lasting exactly n turns.

$$\bar{n} = \sum_{n=1}^{\infty} n \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{n-1}$$

Using the derivative trick we note that $\sum_n nq^{n-1} = \frac{d}{dq}\sum_n q^n$. Also noting that the sum of a geometric series is given by $\sum_n q^n = \frac{1}{1-q}$, we find

$$\sum_{n=1}^{\infty} n \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{n-1} = \frac{1}{6}\cdot\frac{d}{dq}\left(\frac{1}{1-q}\right)_{q=5/6} = \frac{1}{6}\cdot\frac{1}{(1-5/6)^2} = 6$$

1.5 1-D Random Walk

Reif §1.6: Consider the random walk problem with $p = q$ and let $m = n_1 - n_2$ denote the net displacement to the right. After a total number of steps, calculate the following mean values: $\overline{m}$,$\overline{m^2}$, $\overline{m^3}$, $\overline{m^4}$.

From the text (§1.4.1) we have the probability of a number of steps n_1 to the right, and a number of steps $n_2 = N - n_1$ to the left as

$$W(n_1) = \frac{N!}{n_1!(N-n_1)!}p^{n_1}q^{N-n_1}$$

First we note that by the binomial theorem

$$\sum_{n_1=0}^{N} W(n_1) = (p+q)^N = 1$$

Thus we have for any function $f(n_1)$

$$\begin{aligned}\overline{f(n_1)} &= \frac{\sum_{n_1=0}^{N} W(n_1)f(n_1)}{\sum_{n_1=0}^{N} W(n_1)} \\ &= \sum_{n_1=0}^{N} f(n_1)W(n_1)\end{aligned}$$

For the mean displacement $\overline{m}$ we have

$$\begin{aligned}
\overline{m} &= \overline{n_1 - n_2} \\
&= \sum_{n_1=0}^{N} (n_1 - n_2) W(n_1) \\
&= \sum_{n_1=0}^{N} \left(p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q} \right) W(n_1) \\
&= \left(p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q} \right) \sum_{n_1=0}^{N} W(n_1) \\
&= \left(p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q} \right) (p+q)^N
\end{aligned}$$

Taking these derivatives and with $p = q = 1/2$ we see:

$$\overline{m} = pN(p+q)^{N-1} - qN(p+q)^{N-1} = N(p+q)^{N-1}(p-q)$$

which gives $\boxed{\overline{m} = 0.}$

Similarly we have for the mean square displacement $\overline{m^2}$ we have

$$\begin{aligned}
\overline{m^2} &= \sum_{n_1=0}^{N} (n_1 - n_2)^2 W(n_1) \\
&= \left(p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q} \right)^2 \sum_{n_1=0}^{N} W(n_1) \\
&= \left(p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q} \right) (N(p+q)^{n-1}(p-q)) \\
&= pN(p+q)^{N-1} + pN(N-1)(p+q)^{N-2}(p-q) \\
&\qquad - qN(N-1)(p+1)^{N-2}(p-q) + qN(p+q)^{N-1}.
\end{aligned}$$

Simplifying we see the suggestive form:

$$\begin{aligned}
\overline{m^2} &= N(p+q)^N + N(N-1)(p+q)^{N-2}(p-q)^2 \\
&= N\sum W(n_1) + N(N-1)(p+q)^{N-2}(p-q)^2.
\end{aligned}$$

Setting $p = q = 1/2$ gives $\boxed{\overline{m^2} = N}$

For $\overline{m^3}$ we can similarly construct:

$$\begin{aligned}
\overline{m^3} &= \overline{(n_1 - n_2)^3} \\
&= \sum_{n_1=0}^{N} (n_1 - n_2)^3 W(n_1) \\
&= \left(p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q}\right)^3 \sum_{n_1=0}^{N} W(n_1) \\
&= \left(p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q}\right)\overline{m^2} \\
&= \left(p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q}\right)[N\sum W(n_1) + N(N-1)(p+q)^{N-2}(p-q)^2] \\
&= N^2(p+q)^{N-1}(p-q) \\
&\quad +N(N-1)(N-2)(p+q)^{N-3}(p-q)^3 \\
&\quad +2N(N-1)(p+q)^{N-1}(p-q) \\
&= [N + 2(N-1)]\overline{m} + N(N-1)(N-2)(p+q)^{N-3}(p-q)^3
\end{aligned}$$

where we've rewritten the expression in terms of $\overline{m}$ in order to simplify the computation for higher order powers. Applying $p = q = 1/2$ again we see $\boxed{\overline{m^3} = 0}$.

Finally for $\overline{m^4}$ we have

$$\begin{aligned}
\overline{m^4} &= \left(p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q}\right)\overline{m^3} \\
&= \left(p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q}\right)[(3N-2)\overline{m} + N(N-1)(N-2)(p+q)^{N-3}(p-q)^3] \\
&= (3N-2)\overline{m^2} + 3N(N-1)(N-2)(p+q)^{N-2}(p-q)^2 \\
&\quad +N(N-1)(N-2)(N-3)(p+q)^{N-4}(p-q)^4
\end{aligned}$$

Note that we can reduce $\overline{m^4}$ further in terms of the previous mean values by utilizing the relation $N(N-1)(p+q)^{N-2}(p-q)^2 = \overline{m^2} - N\sum W(n_1)$. This makes it easier to determine the mean of higher order powers of the net displacement.

$$\overline{m^4} = (6N-8)\overline{m^2} - (3N-6)N\sum W(n_1) + N(N-1)(N-2)(N-3)(p+q)^{N-4}(p-q)^4$$

Evaluating with $p = q = 1/2$ we get

$$\boxed{\overline{m^4} = (3N-2)N}$$

□

1.6 Alternative Analysis of the 1-D Random Walk

In lecture and in the text, we evaluated the probability distribution for taking n_+ of N total steps in the $+x$ direction $P_N(n_+)$ and, by substituting $m = 2n_+ - N$, the distribution of the net number of steps m in $+x$ direction. Using this distribution, we calculated the mean number of steps and the standard deviation.

Another approach is to consider the probability distribution of the individual steps, as follows. Assume that all steps have the same length l, and that the probability of taking steps in the $+x$ and x directions are p and q, respectively.

(a) Sketch the probability distribution of a single step s_i versus x. Does this correspond to any of the standard probability distributions we have considered so far?

(b) What are the mean and standard deviation of the distribution of s_i?

(c) The total displacement $x_N = ml$ after N steps can be expressed as a sum of N statistically independent random variables s_i. Evaluate the mean number of steps taken in the $+x$ direction. (Hint: What is the mean of a sum of independent random variables?)

(d) Evaluate the standard deviation of m. (Hint: What is the mean of a product of statistically independent random variables?)

(e) Similarly, evaluate the expectation value of m^3 and m^4. Compare your answers with the previous question.

(f) Arguing based upon the Central Limit Theorem, what would you expect the probability distribution of m to look like in the limit of large N, i.e., when you add up a very large number of statistically independent random variables each with the distribution sketched in (a)? What should be the mean and standard deviation of this distribution?

(a) s_i is non-zero for all x except for $x = l$ and $x = ?l$. $s_i(x)$ is a continuous probability distribution, and is non-zero only at two points. Therefore, $s_i(x)$ is a sum of two Delta functions,

$$s_i(x) = p\delta(x - l) + q\delta(x - l).$$

Note that with $p + q = 1$, we have $\int s_i(x)dx = 1$.

(b) The mean is

$$\overline{s_i} = \int_{-\infty}^{\infty} x s_i(x)dx = lp + (-l)q = l(p - q)$$

while mean of the squares is

$$\overline{s_i^2} = \int_{-\infty}^{\infty} x^2 s_i(x)dx = l^2 p + (-l)^2 q = l^2.$$

Therefore the average $l(p - q)$ and the standard deviation is $2l\sqrt{pq}$.

(c) The total displacement is a sum of individual displacements, $x_N = \sum_{i=1}^{N} s_i$. Therefore

$$\bar{m} = \frac{1}{l}\overline{x_N} = \frac{1}{l}\sum_{i=1}^{N}\overline{s_i} = N(p-q).$$

(d) To calculate the standard deviation, we first find, $\overline{x_N^2}$:

$$\overline{x_N^2} = \overline{\left(\sum_{i=1}^{N} s_i\right)\left(\sum_{i=1}^{N} s_i\right)} = \sum_{i,j=1}^{N}\overline{s_i s_j} = \sum_{i\neq j=1}^{N}\overline{s_i s_j} + \sum_{i=1}^{N}\overline{s_i s_i}.$$

Now since individual steps are statistically independent $\overline{s_i s_j} = \overline{s_i}\,\overline{s_j}$ and we get

$$\overline{x_N^2} = \sum_{i\neq j=1}^{N} l^2(p-q)^2 + \sum_{i=1}^{N} l^2 = N(N-1)l^2(p-q)^2 + Nl^2$$

and so $\overline{m^2} = N(N-1)(p-q)^2 + N$ and the standard deviation is

$$\sigma_m = \sqrt{\overline{m^2} - \overline{m}^2} = \sqrt{-N(p-q)^2 + N} = 2\sqrt{Npq}.$$

(e) First we must calculate

$$\overline{s_i^3} = \int_{-\infty}^{\infty} x^3 s_i(x)dx = l^3 p + (-l)^3 q = l^3(p-q).$$

$$\overline{s_i^2} = \int_{-\infty}^{\infty} x^2 s_i(x)dx = l^2 p + (-l)^2 q = l^4.$$

Then, as before, we must separate $s_i s_j s_k$ or $s_i s_j s_k s_p$ into cases where there are one, two, three or four different variables.

$$\overline{x_N^3} = \overline{\left(\sum_{i=1}^{N} s_i\right)^3} = \sum_{i,j,k=1}^{N}\overline{s_i s_j s_k} = \sum_{i\neq j\neq k=1}^{N}\overline{s_i s_j s_k} + 3\sum_{i\neq j=1}^{N}\overline{s_i s_i s_j} + \sum_{i=1}^{N}\overline{s_i s_i s_i}.$$

$$\overline{x_N^4} = \sum_{i\neq j\neq k\neq p=1}^{N}\overline{s_i s_j s_k s_p} + 6\sum_{i\neq j\neq k=1}^{N}\overline{s_i s_i s_j s_k} + 3\sum_{i\neq j=1}^{N}\overline{s_i s_i s_j s_j} + 4\sum_{i\neq j=1}^{N}\overline{s_i s_i s_i s_j} + \sum_{i=1}^{N}\overline{s_i s_i s_i s_i}.$$

leading to

$$\overline{m^3} = N(N-1)(N-2)(p-q)^3 + 3N(N-1)(p-q) + N(p-q)$$

$$\begin{aligned}\overline{m^4} = {} & N(N-1)(N-2)(N-3)(p-q)^4 + 6N(N-1)(N-2)(p-q)^2 \\ & + 3N(N-1) + 4N(p-q)^2 + N.\end{aligned}$$

These are indeed the same results as in the previous question.

(f) $x_N = ml$ is the sum of N statistically independent variables with defined mean and the standard deviation. Then, according to the Central Limit Theorem, the probability distribution of m approaches Normal distribution for large N:

$$P_N(m) = \frac{1}{2\pi N 4pq} e^{-\frac{m^2}{2N4pq}}.$$

1.7 Telephone Problem

Reif §1.15: A set of telephone lines is to be installed so as to connect town A to town B. The town A has 2000 telephones. If each of the telephone users of A were to be guaranteed instant access to make calls to B, 2000 telephone lines would be needed. This would be rather extravagant. Suppose that during the busiest hour of the day each subscriber in A requires, on the average, a telephone connection to B for two minutes, and that these telephone calls are made at random. Find the minimum number M of telephone lines to B which must be installed so that at most only 1 percent of the callers of town A will fail to have immediate access to a telephone line to B. (Suggestion: approximate the distribution by a Gaussian distribution to facilitate the arithmetic.)

At an instant, the probability that some subscriber in A is on call to B is

$$p = \frac{2}{60} = \frac{1}{30}$$

obviously, the subscriber is not on call to B with probability $q = 1 - p$.

Assuming all the subscribers have independent probability of being on the phone, the probability of exactly n subscriber being on the line is given by:

$$W(n) = \frac{N!}{n!(N-n)!} p^n q^{N-n}$$

where the factor in front is for the duplicities of choosing the subscribers on the line, and this in fact is a binomial distribution!

Now, suppose we had m phone lines in service. Then, when more than m subscribers try to make a call at the same time, they will fail to connect. Thus, integrating over the the possiblity of more than m subscribers simultaneously making calls at the same time we get the probability of calls being dropped:

$$P(m) = \sum_{n=m+1}^{N} W(n)$$

This summation is rather complex to solve for a large N, so we use Gaussian approximation which should work well for large N, as in (§1.5.19).

$$\boxed{P(m) = \int_{m+1}^{N} dn \frac{1}{\sqrt{2\pi Np(1-p)}} e^{-\frac{(n-Np)^2}{\sqrt{2\pi Np(1-p)}}}}$$

We want to find a number m for which our probability of dropping calls(P) does not exceed 1 percent. Calculating numerically for some values of m we get

$$\begin{aligned}
P(67) &= 0.4340426842 \\
P(70) &= 0.2946689944 \\
P(75) &= 0.1224884633 \\
P(80) &= 0.0370919232 \\
P(83) &= 0.0154180065 \\
P(84) &= 0.0111930934 \\
P(85) &= 0.0080130872
\end{aligned}$$

Thus, the minimum m value leading to call drop probability less than 1 percent is

$$\boxed{m = 85}$$

□

1.8 3-D Isotropic Scattering

Reif §1.18: A molecule of gas moves equal distances l between collisions with equal probability in any direction. After a total N displacements, what is the mean square displacement $\overline{R^2}$ of the molecule from its starting point?

The total distance moved $\vec{R}$ can be expressed as:

$$\vec{R} = \vec{r_1} + \vec{r_2} + \ldots \vec{r_N}$$

The squared distance $R^2 = \vec{R} \cdot \vec{R}$ thus has mean

$$\begin{aligned}\overline{R^2} &= \overline{r_1^2} + \overline{r_2^2} + \ldots + \overline{r_N^2} + 2\overline{\vec{r_1} \cdot \vec{r_2}} + \ldots + 2\overline{\vec{r_1} \cdot \vec{r_N}} \\ &\quad + \cdots + 2\overline{\vec{r_{N-1}} \cdot \vec{r_N}}\end{aligned}$$

The r_n^2 terms have mean l^2, whereas the cross terms are given by

$$\overline{\vec{r_m} \cdot \vec{r_n}} = \overline{l^2 \cos\theta_{mn}}$$

where θ_{mn} is the angle between the direction of the mth scattering and the direction of the nth scattering. Since θ_{mn} is uniformly distributed in $(0, \pi)$, we see

$$\overline{\vec{r_m} \cdot \vec{r_n}} = l^2 \overline{\cos\theta_{mn}} = 0.$$

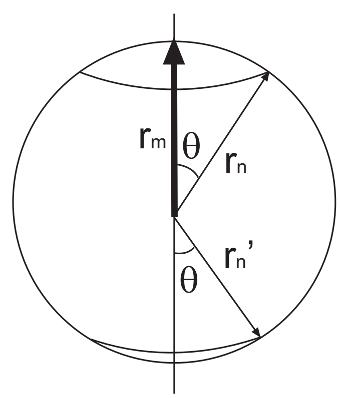

Figure 1: For a given $\vec{r_m}$ we see that the for each possible vector $\vec{r_n}$, there is another, equally probable $\vec{r_n}'$ for which the product $\vec{r_m} \cdot \vec{r_n} = -\vec{r_m} \cdot \vec{r_n}'$, thus when we average over the uniform angular distribution, we see that we must have $\overline{\vec{r_m} \cdot \vec{r_n}} = 0$.

This gives us

$$\boxed{\overline{R^2} = Nl^2}$$

which has a root mean squared value $\sqrt{\overline{R^2}} = \sqrt{N} l$. □

1.9 Uniform Distributions on Circles and Spheres

Reif §1.24:

(a) A particle is equally likely to lie anywhere on the circumference of a circle. Consider as the z-axis any straight line in the plane of the circle and passing through its center. Denote by θ the angle between this z-axis and the straight line connecting the center of hte circle to the particle. What is the probability that this angle lies between θ and $\theta + d\theta$?

(b) A particle is equally likely to lie anywhere on the surface of a sphere. Consider any line through the center of this sphere as the z-axis. Denote by θ the angle between this z-axis and the straight line connecting the center of the sphere to the particle. What is the probability that this angle lies between θ and $\theta + d\theta$?

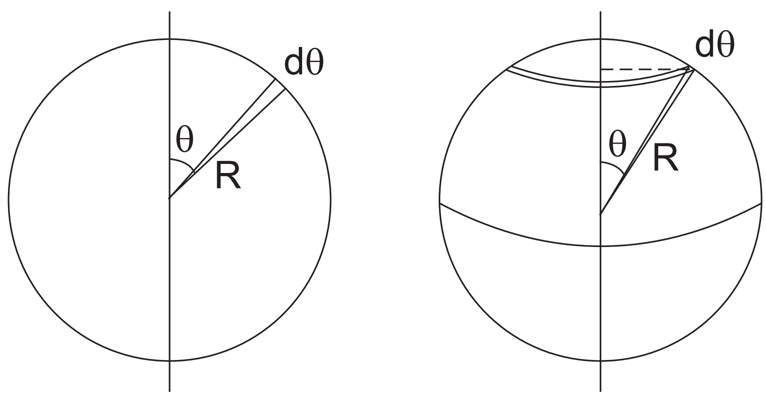

Figure 2: The probability for the particle to be located between θ and $\theta + d\theta$ on a circle and sphere are proportional to the arc length subtended and the area subtended respectively.

(a) Since we have a uniform distribution of probability on the circumference of the circle we have the infinitesimal probability proportional to the infinitesimal arc length.

$$dP \propto dS = Rd\theta$$

where R is the radius of the circle.

Remembering that the probability that the particle is somewhere on the circle must be 1, we see that for some constant C

$$\int dP = C \int Rd\theta = 2\pi RC = 1$$

giving us that $C = 1/2\pi R$.

Thus the probability that the particle will lie between θ and $\theta + d\theta$ is

$$\boxed{p(\theta)d\theta = \frac{Rd\theta}{2\pi R} = \frac{d\theta}{2\pi}}$$

(b) Similarly for the sphere, we see that the probability is proportional to the area

$$dP \propto dA$$

The area covered by particles between θ and $\theta + d\theta$ is given by the area of the ribbon shown, which has width $Rd\theta$ and circumference $2\pi R \sin\theta$. The total surface area of a sphere is of course $4\pi R^2$. Thus we have

$$\begin{aligned} p(\theta)d\theta &= \frac{dA}{4\pi R^2} \\ &= \frac{2\pi R \sin\theta R d\theta}{4\pi R^2} \\ &= \frac{2\pi R^2 \sin\theta d\theta}{4\pi R^2} \end{aligned}$$

giving

$$\boxed{p(\theta)d\theta = \frac{1}{2}\sin\theta d\theta}.$$

Integrating from $\theta = 0$ to π gives a total probability $\int p(\theta)d\theta = 1$.

□

1.10 Waiting Times

(a) The probability density for observing a car is $p(t)dt = dt/\tau$, where $\tau = 5$ min. Hence the average number of cars in the time interval dt is dt/τ. Then, in one hour the average number is

$$\boxed{\frac{60 \text{ min}}{dt}\frac{dt}{\tau} = 12 \text{ cars.}}$$

(b) In a randomly chosen ten minute interval, exactly 2 buses will be seen: $P_{bus}(n) = \delta_{n,2}$. In order to calculate the number of cars arriving in an interval $T = 10$ min, we divide this interval into N shorter intervals of length $dt = T/N$. The probability of observing a car in one of the short intervals is $p = dt/\tau$, and if dt is infinitesimally small, we can ignore the probability of two cars arriving in the same interval. So, we have a binomial probability distribution (N trials, probability p for success, and $1-p$ for failure), so if we take $N \to \infty$, $p \to 0$, as above, we get a Poisson distribution with parameter $\lambda = Np = T/\tau = 2$.

$$\boxed{P_{car}(10 \text{ min}, n) = \frac{2^n}{n!}e^{-2}.}$$

(c) The bus probability is a delta function, thus $\bar{n}_{bus} = 2$, $\text{var}(n_{bus}) = 0$. In this case of the car, rather than calculate the mean and variation of the Poisson distribution, we can just take the limit $N \to \infty$, $p \to 0$, $Np \to T/\tau = 2$ in the binomial distribution. Thus $\boxed{\overline{n}_{car} = Np = 2}$, $\boxed{\text{var}(n_{car}) = Np(1-p) \to 2.}$

(d) Clearly, $p_{bus}(\Delta t) = \delta(\Delta t - 5)$. Thus $\overline{\Delta t} = 5\text{min}$, $\text{var}(\Delta t) = 0$. For the car, we need to calculate the probability density for a time interval $\Delta t - dt$ with no cars at all, and then a time interval dt with just one car. Since these two probabilities are uncorrelated, the joint probability is just the product of the two:

$$p_{car}(\Delta t)dt = P_{car}(\Delta t, n=0)P_{car}(dt, n=1) = \frac{(\Delta t/\tau)^0}{0!}e^{-\Delta t/\tau} \cdot \frac{(dt/\tau)}{1!}e^{-dt/\tau} = \frac{dt}{\tau}e^{-\Delta t/\tau}$$

This is the exponential probability distribution.

$$\boxed{\overline{\Delta t} = \frac{1}{5}\int_0^\infty te^{-\Delta t/5}dt = 5 \text{ min,}} \qquad \boxed{\text{var}(\Delta t) = \frac{1}{5}\int_0^\infty (t-5)^2 e^{-\Delta t/5}dt = 25 \text{ min}^2.}$$

(e) In the case of a bus, since the observer came at a random time, there is a uniform probability density

$$p_{bus}(\Delta t) = \begin{cases} 1/5 & \Delta t < 5 \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{\overline{\Delta t} = \frac{1}{5}\int_0^5 tdt = 2.5 \text{ min,}} \qquad \boxed{\text{var}(\Delta t) = \frac{1}{5}\int_0^5 (t-2.5)^2 dt = 2.08 \text{ min}^2.}$$

In the case of the car, the probability density is the same exponential density that we calculated in (d). The probability of observing a car at any given moment does not depend on what happened before, and therefore $p_{car}(\Delta t)$ does not depend on the starting time of the measurement. Hence $\overline{\Delta t} = 5$ min, $\mathrm{var}(\Delta t) = 25$ min^2.

(f) In a dirty metal or dilute gas, the time between collisions is random. The probability to collide at a certain time interval does not depend on past collisions, and therefore the scenario of exponentially distributed collision times (like the cars) is more relevant.

(g) The probability density of collisions times is exponential

$$\boxed{p(\Delta t) = \frac{dt}{\tau} e^{-\Delta t/\tau}}$$

The mean time between collisions is $\boxed{\overline{\Delta t} = \tau}$.

(h) As we have seen, this process has no memory and therefore

$$\boxed{p(\Delta t_{prev})dt = \frac{dt}{\tau} e^{-\Delta t_{prev}/\tau},} \qquad \boxed{p(\Delta t_{next})dt = \frac{dt}{\tau} e^{\Delta t_{next}/\tau},}$$

and

$$\boxed{\overline{\Delta t_{next}} = \overline{\Delta t_{prev}} = \tau.}$$

(i) From (g) we get that the average time between collisions is τ. In (h), we get that for a randomly chosen time t, the time between the consecutive collisions will be $\overline{\Delta t_{next}} + \overline{\Delta t_{prev}} = 2\tau$. This appears to be a paradox. However, the results are consistent! If an observer arrives at a random time t, she is more likely to fall on one of the long time intervals, and therefore the interval length measured by such an observer would not reflect the actual distribution of collision times.

□