

Solutions to Problem Set 2

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2.1 Classical Particle in a 1-D Box

Reif §2.1: A particle of mass m is free to move in one dimension. Denote its coordinate by x and its momentum by p. Suppose that this particle is confined within a box so as to be located between $x = 0$ and $x = L$, and suppose that its energy is known to lie between E and $E + \delta E$. Draw the classical phase space of this particle, indicating the regions of this space which are accessible to the particle.

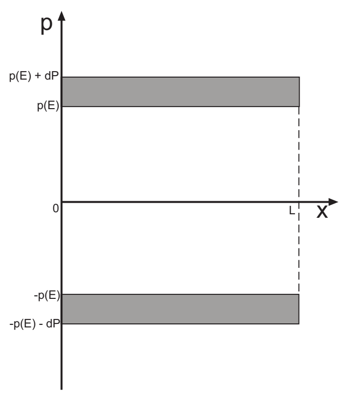

Figure 1: Solution to problem 2.1

Assuming the particle is confined classically to the box $0 \le x \le L$, we have

$$E < \frac{p^2}{2m} < E + \delta E$$

solving for p we define

$$\sqrt{2mE} < p < \sqrt{2m(E + \delta D)} \tag{1}$$

linearizing we see

$$p(E) = \sqrt{2mE} \qquad \delta p = \sqrt{\frac{m}{2E}}\delta E \tag{2}$$

□

2.2 Two Particles in a Box

Reif §2.2: Consider a system consisting of two weakly interacting particles, each of mass m and free to move in one dimension. Denote the respective position coordinates of the two particles by x_1 and x_2, their respective momenta by p_1 and p_2. The particles are confined within a box with end walls located at $x = 0$ and $x = L$. The total energy of the system is known to lie between E and $E + \delta E$. Since it is difficult to draw in four-dimensional phase space, draw seperately the part of phase space involving x_1 and x_2, and that involving p_1 and p_2. Indicate on these diagrams the regions of phase space accessible to the system.

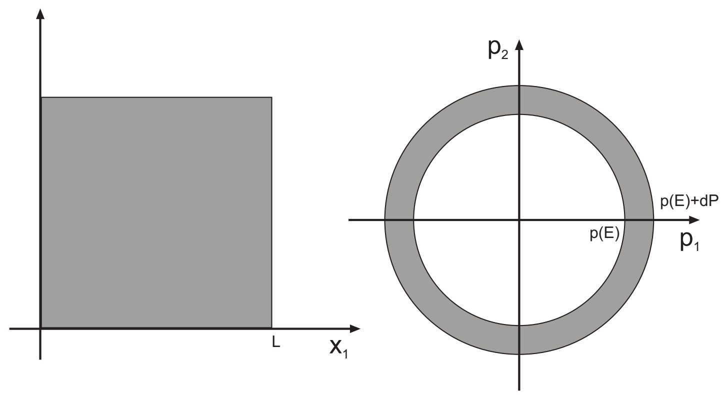

Figure 2: Solution for problem 2.2

Again, the classical particles are bound to the region $0 < x < L$. Due to the weak coupling between the particles, the momenta must obey

$$E < \frac{1}{2m}(p_1^2 + p_2^2) < E + \delta E \tag{3}$$

which bounds the solution between two circles in momentum space,

$$2mE < p_1^2 + p_2^2 < 2m(E + \delta E) \tag{4}$$

with radii $\sqrt{2mE}$ and $\sqrt{2m(E+\delta E)} = \sqrt{2mE} + \sqrt{m/2E}\delta E$. □

2.3 Ensemble of Harmonic Oscillators

Reif §2.3: Consider an ensemble of classical one-dimensional harmonic oscillators.

(a) Let the displacement x of an oscillator as a function of time t be given by $x = A\cos(\omega t + \phi)$. Assume that the phase angel ϕ is equally likely to assume any value in its range $0 < \phi < 2\pi$. The probability $w(\phi)d\phi$ that ϕ lies in the range between ϕ and $\phi + d\phi$ is then simply $w(\phi)d\phi = (2\pi)^{-1}d\phi$. For any fixed time t, find the probability $P(x)dx$ that x lies between x and $x + dx$ by summing $w(\phi)d\phi$ over all angles ϕ for which x lies in this range. Express $P(x)$ in terms of A and x.

(b) Consider the classical phase space for such an ensemble of oscillators, their energy being known to lie in the small range between E and $E + \delta E$. Calculate $P(x)dx$ by taking the ratio of that volume of phase space lying in this energy range *and* in the range between x and $x + dx$ to the total volume of phase space lying in the energy range between E and $E + \delta E$. Express $P(x)$ in terms of E and x. by relating E to the amplitude A, show that the result is the same as that obtained in part (a).

(a) We have

$$\begin{aligned} p(x)dx &= \sum \frac{w(\phi)}{|dx/d\phi|} d\phi \\ &= 2\frac{dx}{2\pi A \sin(\omega t + \phi)} \\ &= \frac{dx}{\pi\sqrt{A^2 - x^2}} \end{aligned}$$

(b) We have the energy as a function of the amplitude:

$$E = \frac{p^2}{2m} + \frac{kx^2}{2} = \frac{kA^2}{2}$$

The equal energy (equal amplitude) contour, in phase space is an ellipse (see Fig 2.3.1 in Reif). If we make the transformation $p'^2 = p^2/(mk)$, we get a circle as the equal energy (amplitude) contour. $A^2 = x^2 + p'^2$. Now, the phase space volume lying between E and $E + \delta E$ is represented by the area of a shell between A and $A + \delta A$ where δA is a function of δE.

$$W(A)\delta A = 2\pi A \delta A.$$

In order to calculate which portion of this shell lies between x and dx, we need to move polar coordinates

$$\cos\theta = \frac{x}{A}, \qquad d\theta = \frac{dx}{A\sin\theta} = \frac{dx}{\sqrt{A^2 - x^2}}.$$

Therefore, the area of the two parts of the shell that lies between x and $x + dx$ is

$$W(x, A)dx\delta A = 2Ad\theta\delta A = \frac{2Adx\delta A}{\sqrt{A^2 - x^2}}.$$

and the probability of being in this interval is

$$p(x)dx = \frac{W(x, A)dx\delta A}{W(A)\delta A} = \frac{dx}{\pi\sqrt{A^2 - x^2}}$$

2.4 Magnetization of Spins

Reif §2.4: Consider an isolated system consisting of a large number N of very weakly interacting localized particles of spin $\frac{1}{2}$. Each particle has a magnetic moment μ which can point either parallel of antiparallel to an applied field H. The energy E of the system is then $E = -(n_1 - n_2)\mu H$, where n_1 is the number of spins aligned parallel to H and n_2 the number of spins aligned antiparallel to H.

(a) Consider the energy range between E and $E + \delta E$ where δE is very small compared to E but is microscopically large so that $\delta E \gg \mu H$. What is the total number of states $\Omega(E)$ lying in this energy range?

(b) Write down an expression for $\ln \Omega(E)$ as a function of E. Simplify this expression by applying Stirling's formula in its simplest form

$$\ln n! \approx n \ln n - n.$$

(c) Assume that the energy E is in a region where $\Omega(E)$ is appreciable, i.e., that it is not close to the extreme possible values $\pm N\mu H$ which it can assume. In this case apply a Gaussian approximation to part (a) to obtain a simple expression for $\Omega(E)$ as a function of E.

(a) This problem is like counting the number of states in a binomial distribution. Note this is only counting the number of states. There are two possible states for each of the spins, i.e. parallel and antiparallel. $E = -(n_1 - n_2)\mu H$ can be written as $E = -(2n_1 - N)\mu H$ using $N = n_1 + n_2$. Thus, by counting the number of states for a specific value of n_1, we can directly relate it to the number of states within the energy range. The number of states for n_1 is

$$\Omega(n_1) = \frac{N!}{n_1!(N - n_1)!}$$

Now, $\Omega(E, E+\delta E)$ can be found by counting the number of n_1 within the energy range $E, E + \delta E$. Since $\delta E \gg 2\mu H$, we can approximate the number to be $|\frac{\delta E}{dE/dn}|$. Since $E = -(2n_1 - N)\mu H$, $\frac{dE}{dn} = -2\mu H$. Thus,

$$\Omega(E, E + \delta E) = \Omega(n_1)|\frac{1}{dE/dn}|\delta E = \frac{N!}{n_1!(N - n_1)!}\frac{\delta E}{2\mu H}$$

where we can substitute $n_1 = \frac{1}{2}(N - \frac{E}{\mu H})$ to get

$$\Omega(E, E + \delta E) = \frac{N!}{(N/2 - E/2\mu H)!(N/2 + E/2\mu H)!}\frac{\delta E}{2\mu H}.$$

(b) Using the result of (a),

$$\ln\Omega(E) = \ln N! - \ln(N/2 - E/2\mu H)! - \ln(N/2 + E/2\mu H)! - \ln 2\mu H$$

apply Stirling's formula to this we get

$$\begin{aligned}
\ln\Omega(E) &= N\ln N - N - \ln 2\mu H - \frac{N - E/\mu H}{2}\ln\frac{N - E/\mu H}{2} + \frac{N - E/\mu H}{2} \\
&\quad -\frac{N + E/\mu H}{2}\ln\frac{N + E/\mu H}{2} + \frac{N + E/\mu H}{2}. \\
&= N\ln N - \ln 2\mu H - \frac{N - E/\mu H}{2}\ln\frac{N - E/\mu H}{2} - \frac{N + E/\mu H}{2}\ln\frac{N + E/\mu H}{2}
\end{aligned}$$

(c) Although this is not a random walk, the formula of the number of states is proportional to the random walk probability with $p = q = \frac{1}{2}$. In particular, if we put a factor of $\Omega = 2^N$ in front of the expression and cancel it by adding a factor of $p^n q^{N-n} = \frac{1}{2^N}$, then our distribution is identical to the binomial distrbution with the additional factor of Ω. Thus, using the Gaussian approximation (§1.6.4) on the result of (a),

$$\begin{aligned}
\Omega(E)dE &= \frac{\Omega}{\sqrt{2\pi}\sigma}e^{-(E-\bar{E})^2/2\sigma^2}dE \\
&= \frac{2^N}{\sqrt{2\pi N}\mu H}e^{-\frac{E^2}{2N(\mu H)^2}}dE
\end{aligned}$$

$\bar{E} = (p-q)Nl = 0$, $\sigma = 2\sqrt{Npq}l = 2\sqrt{N\frac{1}{4}}\mu H = \sqrt{N}\mu H$, and $\Omega = 2^N$ has been used.

We can also show this starting from the result of (b). Assuming E is not close to $\pm N\mu H$, i.e. $\frac{E}{\mu H} \ll N$, we can use $\ln(1-x) \approx -x$ for $x \ll 1$. Then,

$$\begin{aligned}
\ln\frac{N \pm E/\mu H}{2} &= \ln\frac{N}{2}(1 \pm \frac{E}{\mu HN}) \\
&= \ln\frac{N}{2} + \ln(1 \pm \frac{E}{\mu HN}) \\
&\approx \ln\frac{N}{2} \pm \frac{E}{\mu HN}
\end{aligned}$$

Using this in the equation for $\ln\Omega(E)$,

$$\begin{aligned}
\ln\Omega(E) &= N\ln N - \ln 2\mu H - \frac{N - E/\mu H}{2}(\ln\frac{N}{2} - \frac{E}{\mu HN}) - \frac{N + E/\mu H}{2}(\ln\frac{N}{2} + \frac{E}{\mu HN}) \\
&= N\ln 2 - \ln 2\mu H - \frac{1}{2N}(\frac{E}{\mu H})^2 \qquad (5)
\end{aligned}$$

This yields the Gaussian approximation

$$\Omega(E) = \frac{2^N}{2\mu H}e^{-\frac{E^2}{2N(\mu H)^2}}.$$

(Note: Two results have constant factor difference because we used a simple form of Sterling's approximation, thereby dropping some terms in the logarithmic scale.) $\square$

2.5 Wire Under Tension

> **Reif §2.9:** The tension in a wire is increased quasi-statically from F_1 to F_2. If the wire has length L, cross-sectional area A, and Young's modulus Y, calculate the work done.

The force F required to stretch by ΔL a wire with initial length L, cross-sectional area A, and Young's modulus Y is given by

$$\frac{F}{A} = Y\frac{\Delta L}{L} \tag{6}$$

The work done in increasing the force from F_1 to F_2 is

$$W = \int p dV = \int F dx \tag{7}$$

So, changing variables so that the integral runs from ΔL_1 to ΔL_2, with $\Delta L_i = \frac{L}{YA}F_i$, we have

$$\int_{\Delta L_1}^{\Delta L_2} \frac{YA}{L} l dl = \frac{YA}{2L}(\Delta L_2^2 - \Delta L_1^2) = \frac{L}{2YA}(F_2^2 - F_1^2) \tag{8}$$

2.6 Exact and Inexact differentials

Reif §2.6: Consider the infinitesimal quantity

$$(x^2 - y)dx + xdy \equiv đF \tag{9}$$

(a) Is this an exact differential?

(b) Evaluate the integrals $\int đF$ between the points $(1,1)$ and $(2,2)$ along the straight line paths connecting the folllowing points:

$$(1,1) \rightarrow (1,2) \rightarrow (2,2)$$
$$(1,1) \rightarrow (2,1) \rightarrow (2,2)$$
$$(1,1) \rightarrow (2,2)$$

(c) Suppose that both sides of the $đF$ equation are divided by x^2. This yields the quantity $đG = đF/x^2$ Is $đG$ and exact differential?

(d) Evaluate the integral $\int đG$ along the three pahts of part (b).

(a) For something to be an exact differential it must be expressed as

$$đF = A(x,y)dx + B(x,y)dy = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

If this was the case then we would have

$$\frac{\partial}{\partial y}A_F(x,y) = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}B_F(x,y),$$

however we see instead that

$$\begin{aligned}
\frac{\partial A_F}{\partial y} &= \frac{\partial}{\partial y}(x^2 - y) \\
&= -1 \\
\frac{\partial B_F}{\partial x} &= \frac{\partial}{\partial x}x \\
&= 1 \\
\Rightarrow \frac{\partial A_F}{\partial y} &\neq \frac{\partial B_F}{\partial x}
\end{aligned}$$

hence $đF$ cannot be an exact differential.

(b) Evaluating $\int đF$ we see

$$\begin{aligned}
F[(1,1) \rightarrow (1,2) \rightarrow (2,2)] &= 0 + \int_1^2 1dy + \int_1^2 (x^2 - 2)dx + 0 = \frac{4}{3} \\
F[(1,1) \rightarrow (2,1) \rightarrow (2,2)] &= \int_1^2 (x^2 - 1)dx + 0 + \int_1^2 2dy = \frac{10}{3} \\
F[(1,1) \rightarrow (2,2)] &= \int_1^2 (x^2 - x)dx + \int_1^2 ydy = \frac{7}{3}
\end{aligned}$$

(c) We have

$$đG = (1 - \frac{y}{x^2})dx + \frac{1}{x}dy$$

giving

$$\begin{aligned}
\frac{\partial A_G}{\partial y} &= \frac{\partial}{\partial y}(1 - \frac{y}{x^2}) \\
&= -\frac{1}{x^2} \\
\frac{\partial B_G}{\partial x} &= \frac{\partial}{\partial x}\frac{1}{x} \\
&= -\frac{1}{x^2} \\
\Rightarrow \frac{\partial A_G}{\partial y} &= \frac{\partial B_G}{\partial x}
\end{aligned}$$

Hence $đG$ is an exact differential.

(d) Using the definition of $đG$,

$$\begin{aligned}
G[(1,1) \to (1,2) \to (2,2)] &= 0 + \int_1^2 dy + \int_1^2 (1 - 2/x^2)dx + 0 = 1 \\
G[(1,1) \to (2,1) \to (2,2)] &= \int_1^2 (1 - 1/x^2)dx + 0 + \int_1^2 dy/2 = 1 \\
G[(1,1) \to (2,2)] &= \int_1^2 (1 - x/x^2)dx + \int_1^2 dy/y = 1
\end{aligned}$$

The integral $\int đG$ does not vary between the three paths.

2.7 Quantum Particle in a Box

Reif §2.7: Consider a particle confined within a box in the shape of a cube of edges $L_x = L_y = L_z$. The possible energy levels of this particle are then given by

$$E = \frac{\hbar^2}{2m}\pi^2 \left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2} \right).$$

(a) Suppose that the particle is in a given state specified by particular values of the three integers n_x, n_y, n_z. By considering how the energy of this state must change when the length L_x of the box is changed quasistaticallly by a small amount dL_x, show that the force exerted by the particle in this state on a wall perpendicular to the x-axis is given by $F_x = \partial E/\partial L_x$.

(b) Calculate explicitly the force per unit area (or pressure on this wall. By averaging over all possible states, find an expression for the mean pressure on this wall. (Exploit the property that the average values $\overline{n_x^2} = \overline{n_y^2} = \overline{n_z^2}$ must be equal by symmetry). Show that this mean pressure can be very simply expressed in terms of the mean energy $\bar{E}$ of the particle and the volume $V = L_x L_y L_z$ of the box.

(a) We have

$$E = \frac{\hbar^2\pi^2}{2m} \left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2} \right)$$

For a small change in distance δL_x, we have the change in energy

$$\delta E = \frac{\partial E}{\partial L_x}\delta L_x = -2\frac{\hbar^2\pi}{m}\frac{n_x^2}{L_x^3}\delta L_x. \tag{10}$$

Which corresponds to work $F_x \delta L_x = \delta W = -\delta E$. Hence we have

$$F_x = \frac{\partial E}{\partial L_x} = 2\frac{\hbar^2\pi}{m}\frac{n_x^2}{L_x^3}. \tag{11}$$

(b) To calculate the mean value of the pressure we first see pressure is defined as

$$P = \frac{F_x}{A_x} = \frac{F_x}{L_y L_z}$$

the mean value is then

$$\bar{P} = \frac{\bar{F}_x}{L_y L_z} = \frac{\hbar^2\pi^2}{m}\frac{\overline{n_x^2}}{L_x^3}\frac{1}{L_y L_z}$$

To find an expression for $\overline{n_x^2}$ we notice

$$\bar{E} = \frac{\hbar^2\pi^2}{2m} \left(\frac{\overline{n_x^2}}{L^2} + \frac{\overline{n_y^2}}{L^2} + \frac{\overline{n_z^2}}{L^2} \right)$$

with $L = L_x = L_y = L_z$. Noting that by symmetry $\overline{n_x^2} = \overline{n_y^2} = \overline{n_z^2}$,

$$\overline{n_x^2} = \frac{2mL^2\bar{E}}{3\hbar^2\pi^2} \tag{12}$$

Hence we have

$$\bar{P} = \frac{2}{3}\frac{\bar{E}}{V}. \tag{13}$$

where $V = L^3$. $\square$