Cornell University, Physics Department — Fall 2014
PHYS-3341 Statistical Physics — Prof. Itai Cohen

Problem Set 9

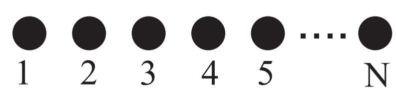
Due Friday Nov 14

9.1 One Dimensional Paramagnet

In this exercise, you are asked to cconsider the Ising model in one dimension,

$$H = -\frac{J}{2}\sum_{i=1}^{N-1}\sigma_i\sigma_{i+1} - \mu_B B \sum_i \sigma_i \ ,$$

where $\sigma_i = \pm 1, i = 1, \dots, N$ (see figure), μ_B is the Bohr magneton, V is the magnetic field, and J is the exchange interaction constant.



For this system the canonical partition function reads

$$Z = \sum_{\sigma_1=\pm 1}\sum_{\sigma_i=\pm 1}\cdots\sum_{\sigma_N=\pm 1}\exp\left[\frac{J}{2kT}(\sigma_1\sigma_2+\sigma_2\sigma_3+\ldots+\sigma_{N-1}\sigma_N)+\frac{\mu_B B}{kT}(\sigma_1+\sigma_2+\ldots+\sigma_N)\right] .$$

Below, you'lll show that in one dimension the $J \neq 0$ Ising model at zero magnetic field can be mapped to the $J = 0$ ising model at nonzero magnetic field. In order to achieve this, instead of using the variables $\sigma_i, i = 1, 2, \dots, N$ to describe the microscopic state of the magnet, one uses variables τ_i which are defined as

$$\sigma_1 = \tau_1, \quad \sigma_2 = \tau_1\tau_2, \quad \sigma_3 = \tau_1\tau_2\tau_3, \quad \dots, \quad \sigma_N = \tau_1\tau_2\dots\tau_N \ .$$

Each variable τ_i can take values ± 1, just like the original variables σ_i.

(a) Express the new variables τ_i in terms of the original variables σ_i.

Hint: Use $\tau_i^2 = \sigma_i^2 = 1$.

(b) Show that the canonical parition function Z with zero magnetic field is formally equi-lvalent to that of the $J = 0$ Ising model with a magnetic field once Z is written in terms of the variables τ_i. Use this to calculate Z.

(c) The same variable change can be used to calculate a "spin-spin correlation function", which is defined as the average $\overline{\sigma_i \sigma_{i+p}}$. Show that spin-spin correlation decays exponentially with the "distance" p,

$$\overline{\sigma_i \sigma_{i+p}} = e^{-|p|/\zeta} \ ,$$

and find an expression for the "correlation length" ζ.

Hint: Use the fact that $\sigma_i \sigma_{i+p} = \tau_{i+1}\tau_{i+2}\ldots\tau_{i+p}$ *if* $p > 0$. *Then argue that the average of a product of* τ_i*'s factorizes.*

9.2 Bosons in a Harmonic Trap

In experiments on dilute cold gases, atoms are confined to a certain region of space by a "magnetic trap", formed by an appropriately configured magnetic field. Close to the center of the trap, the effect of the magnetic field can be described as a harmonic potential, which is described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 r^2$$

where r is the distance from the center of the trap. In the following exercise we consider the case that the atoms in the trap are bosons.

(a) Using quantum mechanics, derive an expression for hte grand canonical partition function $\Xi(\mu, T)$, where μ is the chemical potential and T the temperature of the Bose gas. You may neglect interactions between the atoms.

(b) Derive an expression for the average number of atoms $\bar{N}$, as a function of the chemical potential μ and the temperature T.

(c) Argue that the single-atom ground state will acquire a macroscopic population if the temperature T falls below a critical temperature T_c. This is the Bose-Einstein condensation in a harmonic trap. (*Hint: what is the minimum value of* μ*? How is the ground state occupation different from that of the excited states?*)

(d) Find an expression for the critical temperature T_c for the Bose-Einstein condensation in this system. Sketch the population of the ground state as a function of temperature.

(e) Will there be Bose-Einstein condensation in a one-dimensional harmonic trap? And in a two-dimensional trap? Are these naswers different from the case of an ideal Bose gas confied to a container (i.e., with a confining potential that is zero or infinity, inside or outside the container, repsectively)?

9.3 Velocity Distribution of a Fermi Gas

(a) An ideal Fermi gas is at rest at absolute zero and has a Fermi energy μ. The mass of each particle is m. If v denotes the velocity of a molecule, find $\bar{v}_x$ and $\overline{v_x^2}$.

(b) How would your answer change if the temperature is not zero (but still much smaller than the Fermi temperature)?

9.4 Electron Gas

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9.5 Two particles

Consider a system of two paritcles, each of which can be in any one of threee quantum states of respective energies 0, ϵ, and 3ϵ. The system is in contact with a heat reservoir at temperature T. Calculate the canonical partition function Z

(a) if the two particles are distinguishable;

(b) if the two particles are fermions of the same type;

(c) if the two particles are bosons of the same type.

9.6 Email three questions