Need: dice
Advice Tony Dinsmore Colloquium

Basics of Probabilities & statistics

History: why did people need to come up w/ the theory of probability?

(1501-76) Cardano gives probabilities associated with dice

(1650's) Chevalier de Mere - a gambler - asks Blaise Pascal why some bets make money while others don't.

Pascal ~~[illegible]~~ writes Pierre de Fermat & together they found the theory of probability

Basic Concepts:

What does 30% chance of Rain mean?
What does it mean to say the chance of throwing a double 6 is 1/36?

Need an "ensemble" - Collection of a large number of similarly prepared systems

$$P(\text{event } a) = \frac{\#\text{ of events resulting in } a}{\#\text{ of all possible events}}$$

Ex. Dice

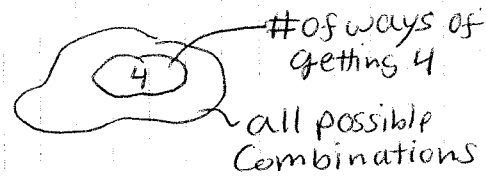
One die $P_1(1) = \frac{1}{6}$

$P_1(2) = \frac{1}{6}$

$\vdots$

two dice $P_2(4) = \sum_y P_1(y)\, P_1(4-y) = \frac{3}{36}$

Pictorially

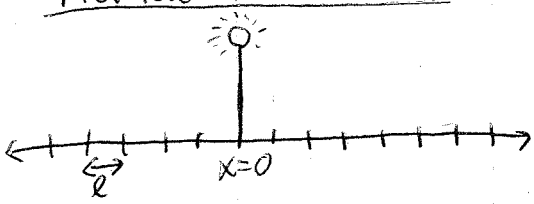


How do we test this?

① Conduct Exp. simultaneously with many systems

② Conduct Exp. over & over with same die

Random walks



Drunk takes steps length ℓ either to the left or to the right

$P(\text{Right}) = p$ $\quad$ $p = q \Rightarrow$ coin, etc

$P(\text{left}) = q$ $\quad$ in general $p \neq q$

at all times $x = m\ell$ where m is an integer

Q: After taking N steps what is probability of being at $x = m\ell$?

Boils down to adding N vectors of equal length but random direction

Important For:

① Polymer Configuration – how big is a chain of atoms in solution
important for manufacturing etc.

② Diffusion of a molecule or Colloidal particle – How fast can you smell your date as he enters a room

③ Magnetism – what is magnetization of N randomly oriented spins

④ Light diffusing through an opaque medium such as milk
DWS Diffusive wave spectroscopy

How do we Calculate $P_N(m)$?

$N = n_1 + n_2$ $\quad$ n_1 = Steps to the right
n_2 = Steps to the left

$m = n_1 - n_2$

$m = 2n_1 - N$ $\quad$ if N is odd m is odd
if N is even m is even

is we have N statistically independent steps with probability p & q

any one sequence will have a probability. Ex: $\underbrace{p \cdot p \cdots p}_{n_1} \cdot \underbrace{q \cdot q \cdots q}_{n_2} = p^{n_1} q^{n_2}$

How many such sequences are there?

Total # of possible permutations for a given outcome:

$\frac{N!}{n_1!\, n_2!}$ ← doesn't care what you pick p or q there are $N!$ combinations
Can exchange all p's with one another
Can exchange all q's with one another

Q: What is the probability of taking N steps ~~to the~~ n_1 to the right & n_2 to the left?

figure this out with your partner

Book uses $W_N(n_1)$

$$P_N(n_1) = \underbrace{\frac{N!}{n_1!\,n_2!}}_{\text{\# of ways of getting this outcome}} \underbrace{p^{n_1} q^{n_2}}_{\text{probability of a single event with desired outcome}}$$

} related to Binomial Distribution $(p+q)^N = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} p^n q^{N-n}$

For $P_N(m)$ need to sub $n_1 = \frac{1}{2}(N+m)$ & $n_2 = \frac{1}{2}(N-m)$

if $p = q = 1/2$

$$P_N(m) = \frac{N!}{\left(\frac{N+m}{2}\right)!\left(\frac{N-m}{2}\right)!}\left(\frac{1}{2}\right)^N$$

Peaked around $m=0$

How do we know this? plug in some #'s

if $m = N$ $\quad \frac{N!}{N!}\left(\frac{1}{2}\right)^N$ $\quad$ if $N=4$ $\quad \left(\frac{1}{2}\right)^4$

if $m = 0$ $\quad \frac{N!}{\frac{N}{2}!\,\frac{N}{2}!}\left(\frac{1}{2}\right)^N$ $\quad \frac{24}{4}\left(\frac{1}{2}\right)^4 = 6\left(\frac{1}{2}\right)^4$

Mean Values

Lets say we have a class that took a midterm. How do we calculate the ave. score?

① Add all the scores & divide by the # of students

② Σ score · Fraction of students that got that score $\Leftarrow$ we almost always use this way

$$\overline{X} = \sum_m X(m) P(m) \qquad X(m) = \ell m \quad \& \quad \sum_m P(m) = 1$$

$$\overline{f} = \sum_m f(m) P(m) \qquad \overline{g} = \sum_m g(m) P(m)$$

$$\overline{f(m) + g(m)} = \sum_m P(m)\left[f(m) + g(m)\right] = \sum_m P(m) f(m) + \sum_m P(m) g(m) = \overline{f(m)} + \overline{g(m)}$$

$$\overline{c\, f(m)} = c\, \overline{f(m)}$$

Lets calculate some things for random walks

$$\overline{n_1} = \sum_{n_1=0}^{N} n_1 P(n_1) = \sum_{n_1=0}^{N} \underbrace{n_1}_{\text{annoying extra part}} \underbrace{\frac{N!}{n_1!(N-n)!} p^{n_1} q^{N-n}}_{\text{Binomial part}}$$

*** Trick to be used over & over:

$$n_1 p^{n_1} = p \frac{\partial}{\partial p}\left(p^{n_1}\right)$$

$$\overline{n_1} = \sum_{n_1=0}^{N} \frac{N!}{n_1!(N-n_1)!} p \frac{\partial}{\partial p}\left(p^{n_1}\right) q^{N-n_1}$$

Switch order of sum & derivative

$$= p \frac{\partial}{\partial p}\left[\sum_{n_1=0}^{N} \frac{N!}{n_1!(N-n_1)!} p^{n_1} q^{N-n_1}\right]$$

Why do we get to do this? Partial over p goes through sum on n_1

$$= p \frac{\partial}{\partial p}(p+q)^N = pN(p+q)^{N-1} = pN$$

Dispersion about the mean

Second Central moment

$$\overline{(\Delta n_1)^2} = \sum_{n_i=1}^{N} P(n_i)(n_i - \overline{n}_1)^2 \geq 0$$

$$\overline{(n_1 - \overline{n}_1)^2} = \overline{(n_1^2 - 2n_1\overline{n}_1 + \overline{n}_1^2)} = \overline{n_1^2} - 2\overline{n}_1\overline{n}_1 + \overline{n}_1^2$$

$$= \overline{n_1^2} - \overline{n}_1^2$$

So we need to calculate $\overline{n_1^2}$ & $\overline{n}_1^2$

Second moment

$$\overline{n_1^2} = \sum_{n_1=0}^{N} \frac{N!}{n_1!(N-n_1)!} p^{n_1} q^{N-n_1} n_1^2$$

use our trick show at home that

$$\overline{n_1^2} = \overline{n}_1^2 + Npq$$

$$\overline{(\Delta n_1)^2} = Npq$$

Root Mean Square deviation $= \left(\overline{\Delta n_1^2}\right)^{1/2}$
or Standard deviation

this quantity tells us about the width of the distribution

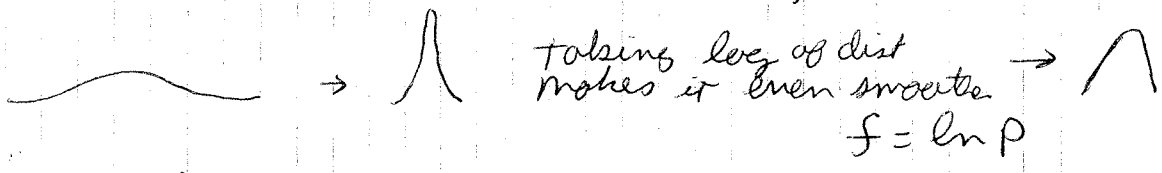
$$\frac{\left(\overline{\Delta n_1^2}\right)^{1/2}}{\overline{n}_1} = \frac{\sqrt{Npq}}{Np} = \sqrt{\frac{q}{p}}\frac{1}{\sqrt{N}} \quad \text{if } p = q \quad \boxed{\left(\overline{\Delta n_1^2}\right)^{1/2} = \frac{1}{\sqrt{N}}\overline{n}_1}$$

as $N \uparrow$ distribution narrows

Central Limit Thm

Outline main idea:

As $N \to \infty$ distribution becomes smooth, continuous, & Narrower



This allows us to use a power law expansion to approximate the distribution. Typically only the first two terms contribute

Taylor expansion about X_0:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3$$

if $f(x_0)$ is a maximum

$f(x_0) = C_1$ Const

$f'(x_0) = 0$

$f''(x_0) = C_2$ Const < 0

$f'''(x_0) = C_3$ small

$\vdots$

if $(x - x_0)$ is small then we can keep only the lowest order term

$$\ln(p) = C_1 + \frac{1}{2} C_2 (x - x_0)^2 + \dots$$

$$p = C_4 e^{\frac{1}{2} C_2 (x - x_0)^2}$$ Gaussian distribution

This is the essence of the Central limit theorem.

Thm. Let X be the sum of N identically and independently distributed random numbers x_i with

$$\overline{x_i} = x_0 \quad \& \quad \overline{x_i^2} - \overline{x_i}^2 = \sigma^2$$

then if $N \to \infty$ the random variable $y \equiv \frac{(X - Nx_0)}{\sigma\sqrt{N}}$ has a Gaussian distribution with zero mean & unit variance i.e.

$$P(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

Any distribution that is smooth & continuous about the maximum & has a finite variance leads to a gaussian distribution at large N.

Following this reasoning:

- $$P(n_1) = (2\pi N p q)^{-1/2} \, \mathrm{Exp}\left[-\frac{(n_1 - Np)^2}{2Npq} \right]$$

or using $\overline{n_1}$ & $\overline{(\Delta n_1^2)}$

- $$P(n_1) = \left[2\pi \overline{(\Delta n_1)^2}\right]^{-1/2} \mathrm{Exp}\left[-\frac{(n_1 - \overline{n_1})^2}{2\overline{(\Delta n_1)^2}} \right]$$

$n_1 = \frac{1}{2}(N+m) \quad m = 2n_1 - N \quad$ so $\Delta m = 2$

- $$P_N(m) = (2\pi N p q)^{-1/2} \, \mathrm{Exp}\left[-\frac{[m - N(p-q)]^2}{8Npq} \right]$$

(Could have done this using Stirlings formula A.6 $\quad n! = \sqrt{2\pi n}\, n^n e^{-n}$ for large n)

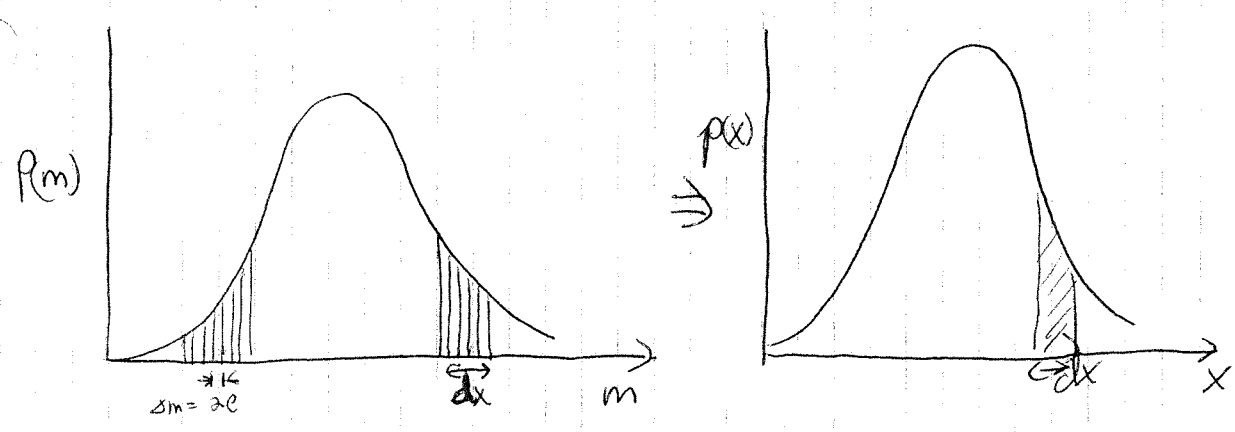
Coarse Graining

Lets now connect all this to real space

What is the probability of being a distance X away in N steps. $\quad X = m\ell \quad$ to coarse grain need $\ell \ll X$ where X is the length of interest in the problem

Ex: Colloidal particles in a microscope min. resolution ~ 0.1 μm

$x \ll$ Scale over which $P(x)$ changes



$$\sum_{m=0}^{N} P(m) = 1 \quad \Rightarrow \quad \int_{-\infty}^{\infty} p(x)\,dx = 1$$

$p(x)$ is the probability density. It is independent of dx. In order to get a probability you need to multiply the probability density by an element of length

$$p(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-(x-\mu)^2/2\sigma^2}\,dx \qquad \mu \equiv (p-q)N\ell \qquad \sigma = 2\sqrt{Npq}\,\ell$$

want to show $\displaystyle\int_{-\infty}^{\infty} p(x)\,dx = 1$

want to calculate $\bar{x}$

$$\bar{x} = \int_{-\infty}^{\infty} x\,p(x)\,dx = \mu$$

want to calculate $\overline{(\Delta x)^2}$

mean square deviation $\quad \overline{(x-\mu)^2} = \displaystyle\int_{-\infty}^{\infty} (x-\mu)^2\,p(x)\,dx$

all of this requires integration of gaussians

Aside: Integrating Gaussians Beautiful trick

$$I \equiv \int_{-\infty}^{\infty} e^{-x^2} dx \qquad I \equiv \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} \underbrace{dx\,dy}_{\text{integrate over } x-y \text{ plane}}$$

switch to polar Coordinates

$$x^2+y^2=r^2 \qquad dx\,dy = r\,dr\,d\theta$$

$$I^2 = \int_0^{\infty}\int_0^{2\pi} e^{-r^2} r\,dr\,d\theta = 2\pi \underbrace{\int_0^{\infty} e^{-r^2} r\,dr}_{\text{trivial}}$$

$$I^2 = 2\pi \int_0^{\infty} -\tfrac{1}{2} d\left(e^{-r^2}\right) = -\pi e^{-r^2}\Big|_0^{\infty} = -\pi(0-1) = \pi$$

$$I = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \quad \text{or} \quad \int_0^{\infty} e^{-x^2} dx = \sqrt{\frac{\pi}{2}}$$

Generally: $I(n) \equiv \int_0^{\infty} e^{-\alpha x^2} x^n dx$

$y = e^{-\alpha x^2}$ $dy = -2\alpha e^{-\alpha x^2} x\,dx$

$$I(0) = \int_0^{\infty} e^{-\alpha x^2} dx = \frac{\sqrt{\pi}}{2\alpha^{1/2}} \qquad I(1) = \int_0^{\infty} e^{-\alpha x^2} x\,dx = \frac{1}{2\alpha}$$

what about higher moments? they can all be related through differentiation to $I(0)$ & $I(1)$ (See appendix A.4)

$$I(2) = -\frac{\partial I(0)}{\partial \alpha} = \frac{\sqrt{\pi}}{4\alpha^{-3/2}} \qquad I(3) = -\frac{\partial I(1)}{\partial \alpha} = \frac{1}{2\alpha^2}$$

$\vdots$

if Limits on even I's are $\int_{-\infty}^{\infty}$ just multiply by 2

$\vdots$

if Limits on odd I's are $\int_{-\infty}^{\infty} = 0$ since integrating an odd function

Back to Probabilities

Changing variables
$y = x - \mu \quad dy = dx$
$\alpha = \frac{1}{2\sigma^2}$, $I(0) \cdot 2$

$$\int_{-\infty}^{\infty} p(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \sqrt{\pi}\,\sqrt{2}\,\sigma = 1 \quad \checkmark$$

Change variables
$y = x - \mu \quad x = y + \mu$
$dy = dx$
$\alpha = \frac{1}{2\sigma^2}$
need $I(0)$;

$$\bar{x} = \int_{-\infty}^{\infty} x\,p(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x\,e^{-(x-\mu)^2/2\sigma^2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \left[\underbrace{\cancel{\int_{-\infty}^{\infty} y\,e^{-y^2/2\sigma^2}\,dy}}_{0} + \mu \int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}\,dy \right]$$

$$\bar{x} = \mu \frac{\sqrt{2\pi}\,\sigma}{\sqrt{2\pi}\,\sigma} \quad \checkmark \qquad \bar{x} = (p-q)N\ell = \bar{m}\,\ell$$

$$\overline{\Delta x^2} = \overline{(x-\mu)^2} = \int_{-\infty}^{\infty} (x-\mu)^2\,p(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} y^2\,e^{-y^2/2\sigma^2}\,dy = \frac{1}{\sqrt{2\pi}\,\sigma} \left[\frac{\sqrt{\pi}}{2} (2\sigma^2)^{3/2} \right]$$

$$\overline{\Delta x^2} = \sigma^2 \quad \checkmark \qquad \overline{\Delta x^2} = 4Npq\ell^2 = \overline{(\Delta m)^2}\,\ell^2$$

For a properly normalized Gaussian distribution you can easily read off $\bar{x}$, $\sqrt{\overline{\Delta x^2}}$

Lets extend this methodology to more than one variable. Ex: Random walk in 2-D, 3-D

	Discrete		Continuous
1-D	$\sum P(m) = 1$	$m \to x$	$\int p(x)\,dx = 1$
2-D	$\sum_{m,n}\sum P(m,n) = 1$	$m \to x$, $n \to y$	$\iint p(x,y)\,dx\,dy = 1$

is x & y directions are not correlated

$$\sum_m P(m) \sum_n P(n) = 1 \qquad \int p(x)\,dx \int p(y)\,dy = 1$$

3-D	$\sum_{mnl} P(m,n,l) = 1$	$m \to x$, $n \to y$, $l \to z$	$\iiint p(x,y,z) = 1$

is x, y, z directions are uncorrelated

$$\sum_m P(m) \sum_n P(n) \sum_l P(l) = 1 \qquad \int p(x)\,dx \int p(y)\,dy \int p(z)\,dz = 1$$

Averages

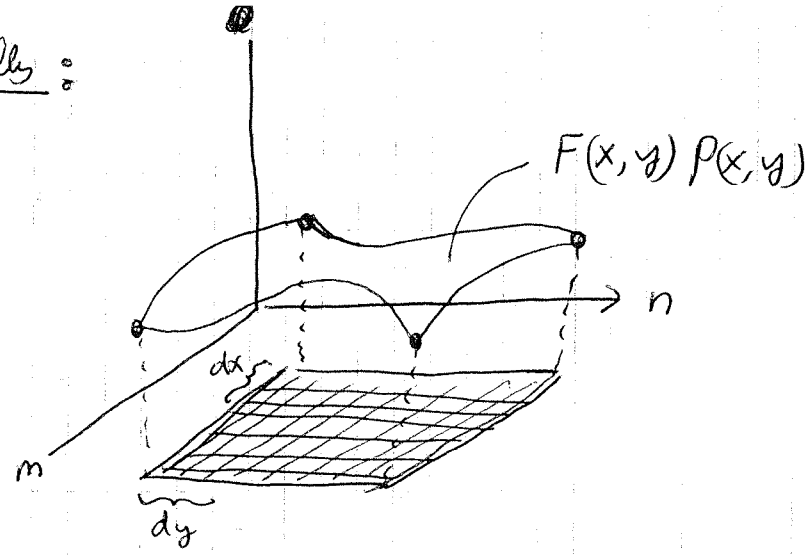
Typically we have some function $F(x_i)$ $i = 1, 2, 3 \ldots$ where average we are trying to evaluate

1-D	$\bar{F} = \sum_m F(m)\,P(m)$	$\bar{F} = \int F\,p(x)\,dx$
2-D	$\bar{F} = \sum_m \sum_n F(m,n)\,P(m,n)$	$\bar{F} = \iint F(x,y)\,p(x,y)\,dx\,dy$
3-D	$\vdots$	$\vdots$
4-D		

$dm \ll dx \ll$ Scale over which Fp changes in x

$dn \ll dy \ll$ Scale over which Fp changes in y

Pictorially:



Summary

When approaching a new problem:

① Determine probability P or probability density ρ

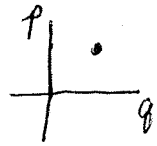
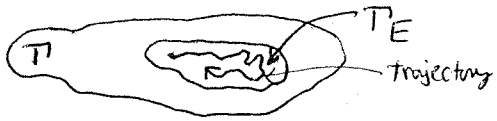
trick: ~~[illegible]~~ Any distribution that dies off rapidly enough, in the limit of large N approaches the Gaussian distribution (all we need to do is normalize, find the mean & std deviation)

② Calculate averages etc. by summing or integrating

For Gaussian distributions, $\overline{x}$, $\sqrt{\overline{\Delta x^2}}$ can be read off immediately

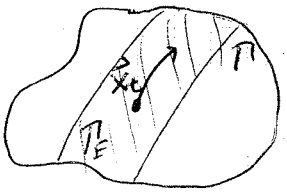
③ Lets extend this methodology to systems of particle

Systems of Particles giving away the punchline

	dice	Classical	QM
a State of system	faces of die showing	$\vec{q}_{1x} \cdots \vec{q}_{n_z}\ \vec{p}_{1x} \cdots \vec{p}_{n_z}$ point in phase space 1 particle 1-D  typically 6n dimensional manifold T  n = # of particles	(a) Spin configuration at lattice sites 2^N different vectors (b) $E_n = (n+\frac{1}{2})\hbar\omega$ Harmonic oscillator n specifies quantum state (c) many H.O. need $n_1 \cdots n_N$ (d) particle in a box need n_x, n_y, n_z (e) Vectors in a Hilbert space $\mathcal{H}$
(b) Statistical ensemble	many throws	many realizations or ensembles (similarly prepared samples) or same sample subjected to the same exp over & over	many realizations or ensembles
c Basic Postulates that need to be verified by exp	any of die faces are equally likely to show	Any point on T is equally probable	Any state in Hilbert space $\mathcal{H}$ is equally likely
d Dynamics	Tossing dice each time (random)	Hamilton's eq's $\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$ if E = Const energy  	Schroedinger Eq. $i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$ + mixing of approximate states by small residual interactions between particles p. 57-58 of Reif
e Probability Calculations & states available to system ex: Constraints due to Energy	Calculate # of ways of getting different results $P = \frac{\#\text{ of realizations}}{\#\text{ of possible outcomes}}$	Complicated systems with many degrees of freedom behave sufficiently "Ergodically" that if we examine a system w/ Energy E at a random time, The Probability of finding it in region R of T_E is $P[R] = \frac{V[R]}{V[T_E]}$ micro canonical ensemble	Subspace $\mathcal{H}_E$ spanned by eigenstates of $\hat{H}$ w/ E values ~~between~~ within some ΔE of E where $\Delta E \ll E$ but ΔE large enough that $\dim \mathcal{H} \gg 1$

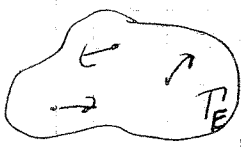
Ergodicity

In Classical mechanics trajectories in phase space are determined by Hamiltonian



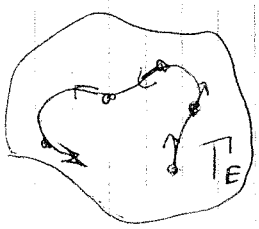
For each point $x \in \Gamma$ (is an element of) define a Hamiltonian "vector field" $\vec{h}$

$$\vec{h} \equiv \left(\underbrace{\frac{\partial H}{\partial p_1}}_{\text{for } \dot{q}_1 \text{ component}}, \frac{\partial H}{\partial p_2} \cdots \frac{\partial H}{\partial p_n} ; \underbrace{-\frac{\partial H}{\partial q_1}}_{\text{for } \dot{p}_1 \text{ component}}, -\frac{\partial H}{\partial q_2}, -\frac{\partial H}{\partial q_n} \right)$$



at each pt. we have a different value (magnitude & direction) for $\vec{h}$

Eq of motion $\dot{\vec{x}} = \vec{h}$ define dynamical trajectories or solutions to Hamiltonian & describe how systems evolve



So what do this curve do?

We know it stays within Γ_E unless H depends explicitly on time

Do they wander around the whole surface?

Are the orbits closed?

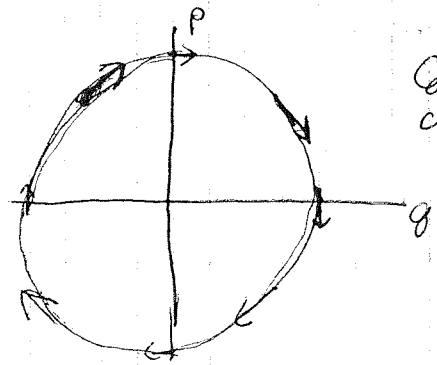
Ex 1 1-D H.O.

dimension of phase space Γ is 2 Coords (q, p)

$H = \frac{1}{2}(p^2 + \omega^2 q^2)$

$\vec{h} = (p, -\omega^2 q)$

ellipses (if $\omega = 1$ it's a circle)



Confined to a 1d surface

dimension of $\Gamma_E = 1$

Ex 2 2 decoupled 1-d H.O.

dimension $\Gamma = 4$ dim phase space (q_1, q_2, p_1, p_2)

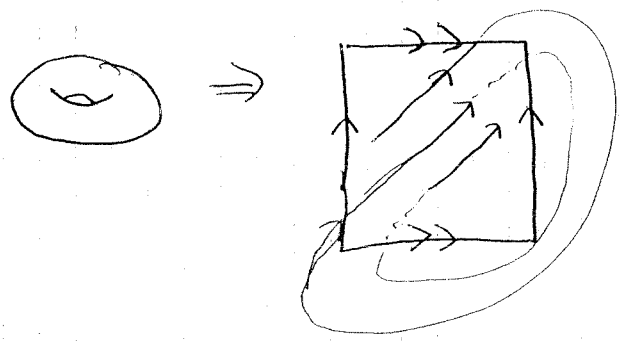
$$H = H_1 + H_2 = \frac{1}{2}\left(p_1^2 + p_2^2 + \omega_1^2 q_1^2 + \omega_2^2 q_2^2\right)$$

Orbits all lie on a 4D surface

On the surface H_1 & H_2 are constant

family of ellipses one for H_1 & one for H_2

Taurus



$\dim \Gamma_E = 2$

If ratio of frequencies is rational then orbits are closed. If ratio of frequencies is not rational then orbits are not closed & each orbit densely fills up the entire 2-taurus.

The path gets arbitrarily close to any point we choose.

Note: We do not cover the space at all — amount of area we cover is zero — lines don't have any thickness

Ergodic Behavior

Boltzman coined this word
Ergode: from Greek literally means work + way

Expect complicated systems w/ many degrees of freedom (with a suitably finite phase space Γ_E finite) to undergo some type of ergodic behavior in that dynamical orbits on Γ_E will "fill up" Γ_E

Lets discuss the possible types of ergodic behavior
These are not necessarily true of real systems

① Quasi Ergodic Behavior

"Almost" all orbits "densely fill up" the whole volume Γ_E

[except for a set of measure zero]

let $A_{y,\varepsilon} = \{ X \in \Gamma_E \mid$ dynamical trajectories through X fails to enter a ball of radius ε around $y \}$

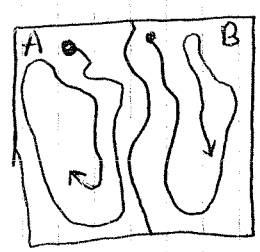
Then $\mathcal{V}[A_{y,\varepsilon}] = 0$ for all $\varepsilon > 0$, all y

② Metric Indecomposability:

Γ_E Cannot be written as a disjoint union of 2 Subsets A, B such that $\mathcal{V}[A] > 0$, $\mathcal{V}[B] > 0$ &

$\underbrace{A_t = A}$, $\underbrace{B_t = B}$

orbits stay on their subsets

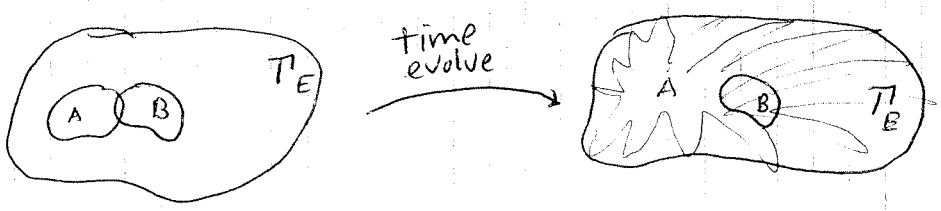


Cannot divide all possible orbits into a number of subsets & just add them up

③ Mixing Behavior

(Sugar ~~dissolving~~ diffusing in water, Hard sphere gas, Collards all obey this)

For any $\underbrace{A, B}_{\text{Subsets}} \subset \Gamma_E$ $\lim_{t \to \infty} \mathcal{V}[A_t \cap B] = \frac{\mathcal{V}[A]}{\mathcal{V}[\Gamma_E]} \cdot \mathcal{V}[B]$



A "uniformly" fills Γ_E But still occupies the same volume $\mathcal{V}[A]$ due to Liouville's Theorem

④ Strict Ergodic Behavior

Every trajectory passes through every point on Γ_E

This is what Boltzmann postulated

This Condition is impossible since $\mathcal{V}[\text{traj}] = 0$

Definitions are related to one another

(if 3 then 2)

- 3 implies 2 : Since if A spreads over T_E uniformly then it must bleed into B as well

(if not 2 then not 3)

- if 2 fails [redacted] 3 fails : if [redacted] we define sets A & B that fail to satisfy 2 then A & B would also fail to satisfy 3

(if not 2 then not 1)

- if 2 fails [redacted] 1 fails : **Since** if we could separate A & B we could find a small "Ball of radius ε" inside B such that a path starting at some point on A does not enter this Ball

(if not 1 then not 2)

- if 1 fails then 2 fails : Since we could let $A = A_{y,\varepsilon}$ & $B = T_E - A_{y,\varepsilon}$

$\boxed{2 \Leftrightarrow 1 \text{ imply one another}}$ I think this is true but not sure

$\boxed{\text{assign as HW}}$

Q: Does taurus example (irrational ratios) satisfy all three definitions? (work this out in groups)

A: No, it violates definition #3 since if we pick a strip in phase space & evolve it in time it moves uniformly without mixing

$$\lim_{t\to\infty} \mathcal{V}[A_t \cap B] \neq \frac{\mathcal{V}[A]\,\mathcal{V}[B]}{\mathcal{V}[T_E]}$$

These three definitions give a flavour of what we mean by ergodic behavior. Research is still going on to determine what type of ergodicity is sufficient for making statistical mechanics works.

Berkhoff's Theorem

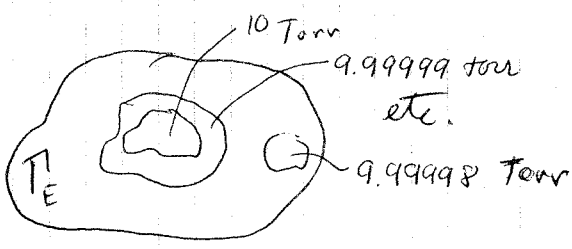
$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(X_{t'})\,dt' = \overline{f(x)}$$

(assumes metric indecomposability)

If system traverses the space ergodically then time average is equal to the space average.

In particular let f be some quantity associated with the system pressure, temperature etc. & Lets Coarse grain the system.

Ex: a wall in a room filled with gas has some pressure as measured by an instrument on it. All the states of the gas that produce that pressure range can be grouped together



Basic Hypothesis of stat Mech:

Complicated systems with many degrees of freedom behave sufficiently "ergodically" that if we examin the system w/ energy E at a random time the probability of finding it in region R of Γ_E is given by

$$P[R] = \frac{\mathcal{V}[R]}{\mathcal{V}[\Gamma_E]}$$

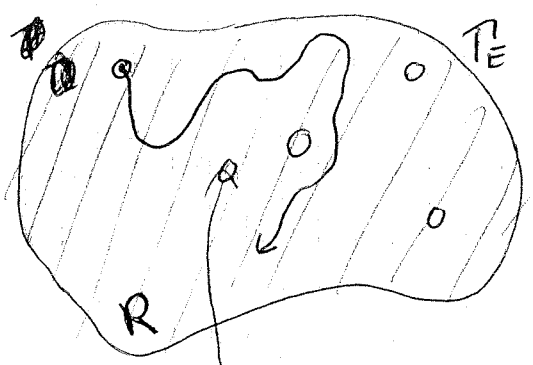
this probability distribution is called the "microcanonical ensemble"

each point in Γ_E is equally likely to be visited by any given trajectory through phase space

Probability in Practice: The real reason stat mech works is that the collection of coarse grained observables $\{F_1, \ldots, F_k\}$ that we measure & care about have the property that the region R of Γ_E where these observables take on their most probable values is such that

$$\mathcal{V}[R] \approx \mathcal{V}[\Gamma_E]$$

The most probable region occupies nearly all of phase space



Fluctuations away from Thermal Equilibrium where other pressures, Temps etc. are possible

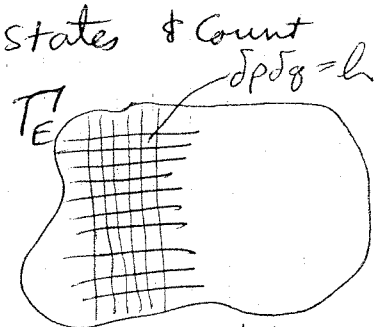
This is the notion of thermal Equil

Even if we start with the system in one of the improbable states, it will enter R very quickly

$$F_i\Big|_{\text{thermal equil}} \equiv \text{The most probable value of } F_i(t) \approx \langle F_i \rangle$$

We could also speak about discrete states & count

if we have a phase-space Γ_E & we want to do some counting of states we can divide Γ_E into N dim cubes of size h^{3N} where $h \equiv \delta p \delta q$ for units of momentum.



total # of states $= \Omega$

Cell size cannot be $\ll h^{3N}$ otherwise you violate the uncertainty principle. Cell size ~~cannot~~ should not be $\gg h^{3N}$ otherwise you loose information

of course actual cell size will never enter into our calculations of physical quantities