Magnetization   Non interacting spins

Section 6.3   Paramagnetism for spin ½ atoms

Section 7.8   "   " Spin J particles

We will do simpler case & where appropriate show how the solution becomes more complicated for general spins.

For each atom in a magnetic field $\epsilon = -\underbrace{\vec{\mu}}_{\text{magnetic moment}} \cdot \vec{H}$ $\vec{\mu} = \underbrace{g}_{g\text{ factor}} \mu_0 \vec{S}$ $\mu_0 = \frac{e\hbar}{2mc}$ } bohr magneton

For an isotropic spin system w/ no neighbor interactions

~~spins will point in the direction of~~ & for $\vec{H} = H_z$

$$\epsilon = -g\mu_0 H S_z \qquad S_z = \pm 1/2$$

$$Z = \sum_{S_z = \pm 1/2} e^{+\beta g \mu_0 H S_z}$$

$$\bar{\mu}_H = \mu \left[\frac{e^{\eta} - e^{-\eta}}{e^{\eta} + e^{-\eta}} \right]$$

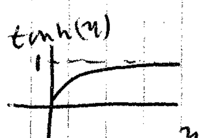
$\eta \equiv \beta \mu H \qquad \mu = |\vec{\mu}|$

note that μ for spin $-1/2$ is $-|\mu|$

μ " " $+1/2$ is $+|\mu|$

$$\bar{\mu}_H = \mu \tanh \frac{\mu H}{kT}$$

for a single atom



Could have also used $\boxed{\bar{\mu}_H = \frac{1}{\beta} \frac{\partial \ln Z}{\partial H}}$

the magnetization $M_0 = N_0 \bar{\mu}_H$

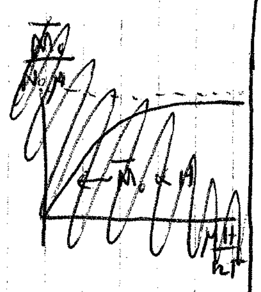
Limits

if $\eta \ll 1$ high T

$\tanh(\eta) = \eta$

if $\eta \gg 1$ low T

$\tanh(\eta) = 1$



Clearly for atoms w/ larger spins this sum is more complicated

$$Z_a = \sum_{m=-J}^{J} e^{\beta g \mu_0 H m} = \frac{e^{-\beta g \mu_0 H J} - e^{\beta g \mu_0 H (J+1)}}{1 - e^{\beta g \mu_0 H}}$$

$\eta \equiv \beta g \mu_0 H$

recall $\frac{1}{1-e^{\eta}} = 1 + e^{\eta} + e^{2\eta} + e^{3\eta} + \dots$

$$e^{-\eta J}[1 + e^{\eta} + e^{2\eta} + \dots] = e^{-\eta J} + e^{-\eta(J-1)} + \dots e^{\eta J} + e^{\eta(J+1)} + e^{\eta(J+2)} + \dots$$

$$e^{+\eta(J+1)}[1 + e^{\eta} + e^{2\eta} + \dots] \qquad e^{\eta(J+1)} + e^{\eta(J+2)} + \dots$$

$$\bar{\mu}_H = \frac{1}{\beta} \frac{\partial \ln Z_a}{\partial H} = g\mu_0 J B_J(\eta)$$

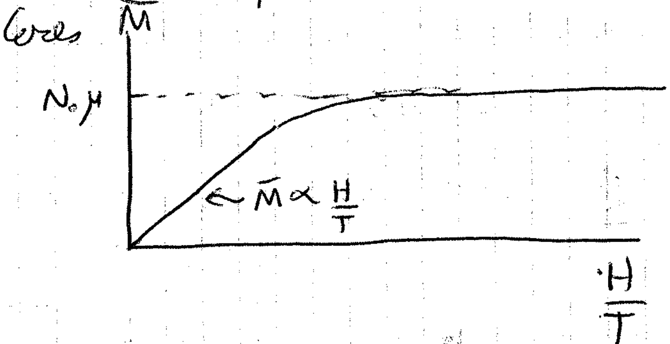
$$B_J(\eta) \equiv \frac{1}{J} \left[\left(J + \tfrac{1}{2}\right) \coth\left(J + \tfrac{1}{2}\right)\eta - \tfrac{1}{2} \coth \tfrac{1}{2}\eta \right]$$

$$M = N_0 \bar{\mu}_H$$

more generally: $\bar{M}_0 = \chi H$ $\quad \chi \equiv$ magnetic susceptibility

	Spin ½	Spin J
for $\frac{\mu H}{kT} \ll 1$	$\chi = \frac{N_0 \mu^2}{kT}$ $\chi \propto T^{-1}$ is Curie's law	$\chi = \frac{N_0 g^2 \mu_0^2 J(J+1)}{3kT}$ $\chi \propto T^{-1}$ prefactor depends on J
for $\frac{\mu H}{kT} \gg 1$	$\bar{M}_0 \rightarrow N_0 \mu$	$\bar{M} \Rightarrow N_0 g \mu_0 J$

~~General~~ For both cases



J can come from unpaired e^- or from the atomic nucleus.

notice however that $\mu_0 = \frac{e\hbar}{2mc}$ ~~Changes drastically~~ decreases by a factor of ~ 1000

for nucleus due to mass term.

So you need to go to temps ~1000 times smaller to achieve nuclear spin orientation

---

Ising model: interactions w/ spins

Adviser: Wilhelm Lenz (1888 - 1957)

named after Ernst Ising who solved model in 1-D.
he saw no phase transition & concluded that this model has no phase transition in any dimension (wrong)

Lars Onsager Solved 2D Ising model in 2D at the Critical pt in 1944

3D problem still an active area of research!

include nearest neighbor interactions

$$\mathcal{H}_{tot} = \mathcal{H}_0 + \mathcal{H}_{jk} = \underbrace{-g\mu_0 H_0 \sum_{j=1}^{N} S_{jz}}_{\text{same as before}} - 2J\sum_{\substack{ij \\ \text{nearest} \\ \text{neighbors}}} \vec{S}_j \cdot \vec{S}_k$$ } Heisenberg Hamiltonian

Simplification: lets only worry about z component

$$\mathcal{H}_{tot} = -g\mu_0 H_0 \sum_{j=1}^{N} S_{jz} - \underbrace{\tfrac{1}{2}}_{\text{so we don't double count}}\left(-2J\sum_{j=1}^{N}\sum_{k=1}^{n} S_{jz}S_{kz}\right)$$ } Ising Hamiltonian

1-D solved by Ising
2-D solved by Onsager

we will solve using a <u>mean field approximation</u>

Replace sum over nearest neighbors w/ mean value of spin (exact in ∞ dimensions)

for jth atom

$$\mathcal{H}_{tot_j} = -g\mu_0 H_0 S_{jz} - 2J S_{jz} \sum_{k=1}^{n} S_{kz}$$

$$2J\sum_{k=1}^{n} \overline{S_{kz}} \equiv g\mu_0 H_m$$

neighbors form an effective H field H_m that still needs to be determined

$$\mathcal{H}_{tot_j} = -g\mu_0 (H_0 + H_m) S_{jz}$$

now we are back to our old solution

$$\overline{S_{jz}} = S\, B_s(\eta) \qquad \text{where } \eta = \beta g\mu_0 (H_0 + H_m)$$

Nothing distinguishes the jth atom from any of its neighboring atoms so all the neighbors atoms should have the same value for $\overline{S}$

So:
$$g\mu_0 H_m = \underbrace{2J}_{\text{coupling const}}\ \underbrace{n}_{\#\text{ of nearest neighbors}}\ \underbrace{S B_s(\eta)}_{\overline{S_{jz}}}$$

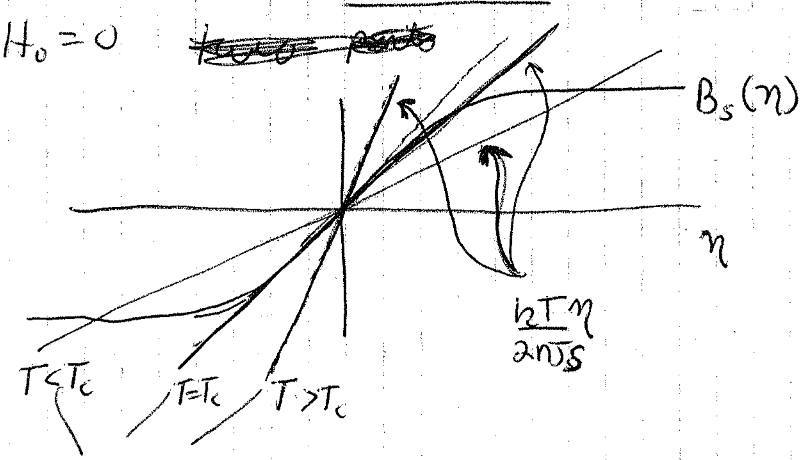
This determines H_m & closes the problem

$$\eta = \beta g\mu_0 (H_0 + H_m) \qquad H_m = \frac{\eta}{\beta g\mu_0} - H_0 \qquad \text{Substitute}$$

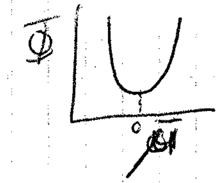
$$\frac{kT}{2nJS}\left(\eta - \frac{g\mu_0 H_0}{kT}\right) = B_s(\eta)$$

Solutions

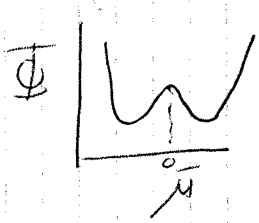
$H_0 = 0$ ~~two parts~~



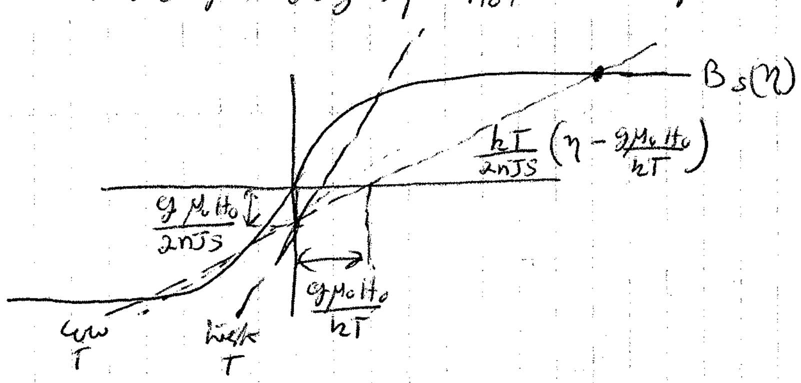
for $T > T_c$ only one solution

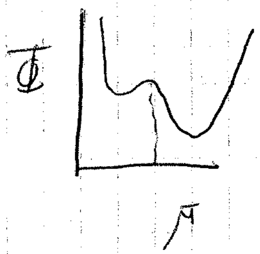


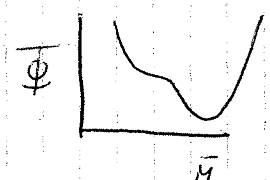
for $T < T_c$



more generally if $H_0 \neq 0$ symmetry breaking field







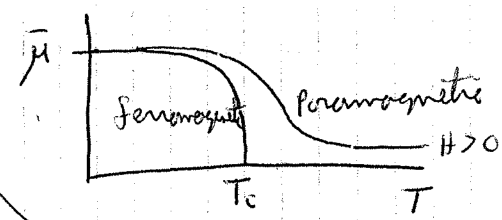
for large enough H

$$H_j = -g\mu_0(H_0 + H_m)S_{jz}$$

spin $\frac{1}{2}$

$$Z = \sum_{j=-\frac{1}{2}}^{\frac{1}{2}} e^{-\beta g\mu_0 \underbrace{(H_0+H_m)}_{} S_{jz}}$$

Bess in Boules



$$e^x + e^{-x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} +$$

$$\ln(e^x - e^{-x}) = \ln\left[2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\right] = \ln 2 + 1 + \underbrace{\frac{x^2}{2} + \frac{1}{2}\frac{x^4}{4!}}_{} + \cdots$$

for $x \ll 1$

only get even terms for the free energy

Since $F = -kT\ln Z$ we get only even terms in our free energy. Back to Landau theory.

~~Reading: Ising model can be mapped onto other phase transitions~~ Ising model

- mean field predicts paramagnetic & ferromagnetic phases
- T_c above which only paramagnetic phase is possible

Lets tie up the story w/ the rubber bands

rubber band on lips — extend rubber band $T\uparrow$; retract rubber band $T\downarrow$

process is adiabatic $\quad dQ = 0 = TdS$

$Q = ST = \text{Const}$

if $S\downarrow T\uparrow$

if $S\uparrow T\downarrow$ ✓

What about exp where we heat rubber band?
Slow demo

recall: $P(\vec{r}) = \frac{1}{\pi N a^2} e^{-r^2/Na^2}$ $\quad$ $\#\text{ of states} = \frac{z^N}{\pi N a^2} e^{-r^2/Na^2}$ $\quad$ # of nearest neighbors

$Z = \sum_r e^{-\beta E_r}$ $\quad$ E is the same for each state

$Z = \frac{z^N e^{-r^2/Na^2}}{\pi N a^2} e^{-\beta E}$

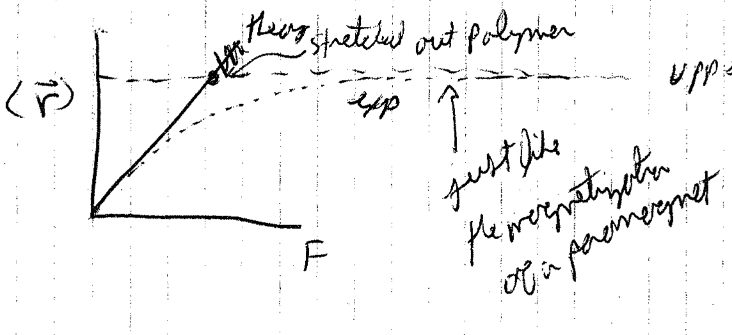
$F = -kT \ln Z = E - kT \ln\left(\frac{z^N e^{-r^2/Na^2}}{\pi N a^2}\right)$

$F = E + \text{Const} + \frac{kT}{Na^2} r^2$ $\quad$ This is just a spring!

$F = \frac{1}{2} K r^2 + \text{Const}$

$K = \frac{kT}{Na^2}$ $\quad$ for a mass on this spring

if $T\uparrow$ $\quad K\uparrow$ & $r\downarrow$ ✓



Quantum Statistics

For indistinguishable particles wavefunction need to be:

* Symmetric for Bosons (integer spin)
* AntiSymmetric for Fermions (spin $n+\frac{1}{2}$, n integer)

This symmetry requirement changes the statistical average over microstates

Ex: two particles can be in two states w/ energy ϵ_1, ϵ_2 (no interaction between particles)

1) distinguishable particles can have 4 states

$$|\epsilon_1\rangle|\epsilon_1\rangle \quad \text{energy } 2\epsilon_1$$

$$\left.\begin{array}{l}|\epsilon_1\rangle|\epsilon_2\rangle \\ |\epsilon_2\rangle|\epsilon_1\rangle\end{array}\right\} \text{energy } \epsilon_1+\epsilon_2 \qquad Z = e^{-2\beta\epsilon_1} + 2e^{-\beta(\epsilon_1+\epsilon_2)} + e^{-2\beta\epsilon_2}$$

$$|\epsilon_2\rangle|\epsilon_2\rangle \quad \text{energy } 2\epsilon_2$$

first particle, second particle

2) For Bosons possible states are:

$$|\epsilon_1\rangle|\epsilon_1\rangle \quad \text{energy } 2\epsilon_1$$

$$\frac{1}{\sqrt{2}}\left(|\epsilon_1\rangle|\epsilon_2\rangle + |\epsilon_2\rangle|\epsilon_1\rangle\right) \quad \text{energy } \epsilon_1+\epsilon_2 \qquad Z = e^{-2\beta\epsilon_1} + e^{-\beta(\epsilon_1+\epsilon_2)} + e^{-2\beta\epsilon_2}$$

$$|\epsilon_2\rangle|\epsilon_2\rangle \quad \text{energy } 2\epsilon_2$$

3) For Fermions

$$\frac{1}{\sqrt{2}}\left(|\epsilon_1\rangle|\epsilon_2\rangle - |\epsilon_2\rangle|\epsilon_1\rangle\right) \quad \text{energy } \epsilon_1+\epsilon_2 \qquad Z = e^{-\beta(\epsilon_1+\epsilon_2)}$$

Physical Implications: ex: mean energy $\bar{E} = -\frac{\partial}{\partial\beta}\ln Z$

probability that two particles occupy the same level:

Classical $$P = \frac{e^{-2\beta\epsilon_1} + e^{-2\beta\epsilon_2}}{e^{-2\beta\epsilon_1} + 2e^{-\beta(\epsilon_1+\epsilon_2)} + e^{-2\beta\epsilon_2}} \rightarrow \frac{1}{2} \text{ if } T\to\infty$$

Bosons $$P = \frac{e^{-2\beta\epsilon_1} + e^{-2\beta\epsilon_2}}{e^{-2\beta\epsilon_1} + e^{-\beta(\epsilon_1+\epsilon_2)} + e^{-2\beta\epsilon_2}} \rightarrow \frac{2}{3} \text{ for } T\to\infty$$

Fermions $P = 0$

As $T \to 0$ For Bosons all particles in ground state
For Fermions still have particles at high energy states since they can't all go down to lower energy levels

Photon Statistics

No restriction on particle # so we use the Canonical dist

# of photons in each energy level ϵ_s is n_s

$$\bar{n}_s = \frac{\sum_{n_1}\sum_{n_2}\cdots n_s e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots n_s\epsilon_s + \cdots)}}{\sum_{n_1}\sum_{n_2} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots + n_s\epsilon_s + \cdots)}}$$

$$\bar{n}_s = \frac{\sum_{n_s} n_s e^{-\beta n_s \epsilon_s}}{\sum e^{-\beta n_s \epsilon_s}} = \frac{-\frac{1}{\beta}\frac{\partial}{\partial \epsilon_s}\sum e^{-\beta n_s \epsilon_s}}{\sum e^{-\beta n_s\epsilon_s}}$$

$$= -\frac{1}{\beta}\frac{\partial}{\partial\epsilon_s}\ln\left(\sum e^{-\beta n_s\epsilon_s}\right)$$

$$\sum_{n_s=0}^{\infty} e^{-\beta n_s\epsilon_s} = 1 + e^{-\beta\epsilon_s} + e^{-2\beta\epsilon_s} + \cdots = \frac{1}{1-e^{-\beta\epsilon_s}}$$ geometric series expansion

$$\bar{n}_s = \frac{1}{\beta}\frac{\partial}{\partial\epsilon_s}\ln\left(1-e^{-\beta\epsilon_s}\right) = \frac{e^{-\beta\epsilon_s}}{1-e^{-\beta\epsilon_s}} = \frac{1}{e^{\beta\epsilon_s}-1}$$

this is the Planck dist

notice that here the sums can go from $n=0 \to \infty$ i.e. no restriction

things get more complicated for particles w/ mass where we typically do have a restriction on the # of particles/level since $\sum_r n_r = N$

For A single particle w/ energy levels ϵ_i

$$H = \sum_i n_i\epsilon_i \qquad N = \sum_i n_i$$

$$Z = \sum_{\{n_1\}}\sum_{\{n_2\}}\cdots e^{-\beta(\epsilon_1 n_1 + \epsilon_2 n_2 + \cdots)}$$

where $\{n_i\}$ means that we sum over all allowed # of particles subject to the restriction that $\sum_i n_i = N$

this is hard

Aside: Trick we used in Section 6.7

Isolated System $\boxed{E}$ use the microcanonical ensemble

alternatively we could have used the canonical ensemble w/ $\bar{E} = E$ as our constraint ~~...~~

~~...~~

In the latter scenario all the calculations for the average quantities work out fine but the calculations for the fluctuations do not!

The advantage is that we get around the counting problem of only including states w/ Energy $E < E_r < E + \delta E$

$$E = \bar{E} = \frac{\sum_r E_r e^{-\beta E_r}}{\sum_r e^{-\beta E_r}}$$

$\Rightarrow$ set this equal to the energy of the system E & β can be determined from the constraint

We can use the same trick for particle #

$$N = \bar{N} = \frac{\sum_r e^{-\beta(E_r - \mu N_r)} N_r}{\sum_r e^{-\beta(E_r - \mu N_r)}}$$

we set $\bar{N} = N$ the total # of particles in our system & now μ can be determined from the constraint

this allows us to sum over all particles #/state w/out the restriction we were running up against previously

Back to Bosen statistics lets use our trick & calculate Ξ rather than Z

$$\Xi_{BE} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_i=0}^{\infty} \cdots e^{-\beta \sum_i n_i \epsilon_i + \beta\mu \sum_i n_i} \qquad \mu = -\frac{\alpha}{\beta}$$

$$\Xi_{BE} = \prod_{i=1}^{\infty} \left(\sum_{n_i=0}^{\infty} e^{-\beta(\epsilon_i - \mu) n_i} \right) = \prod_{i=1}^{\infty} \left(\frac{1}{1 - e^{-\beta(\epsilon_i - \mu)}} \right)$$

using $\frac{1}{1-e^{a}} = \sum_{i=0}^{\infty} e^{-ai}$

$$\ln \Xi_{BE} = -\sum_{i=1}^{\infty} \ln\left(1 - e^{-\beta(\epsilon_i - \mu)}\right)$$

$$\bar{N} = kT \frac{\partial \ln \Xi_{BE}}{\partial \mu} = \sum_{i=1}^{\infty} \frac{e^{-\beta(\epsilon_i - \mu)}}{1 - e^{-\beta(\epsilon_i - \mu)}} = \sum_{i=1}^{\infty} \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}$$

Setting $\bar{N} = N$ allows us to calculate μ

$$\bar{E} = -\frac{\partial}{\partial \beta} \ln \Xi_{BE} = \sum_{i=1}^{\infty} \frac{\epsilon_i}{e^{\beta(\epsilon_i - \mu)} - 1}$$

Setting $\bar{E} = E$ allows us to calculate β

The probability for a given "microstate" characterized by the occupants $n_1, n_2, \ldots$ is

$$P(n_1, n_2, \ldots) = \frac{e^{-\beta \sum_i (\epsilon_i - \mu) n_i}}{\Xi_{BE}}$$

Q: What is the average # of Bosons in a single particle state i?

$$\bar{n}_i = \sum_{\{n_i\}} n_i \, P(n_1, n_2, \ldots, n_i, \ldots)$$

$$= \frac{\sum_{\{n_i\}} n_i \, e^{-\beta \sum_i (\epsilon_i - \mu) n_i}}{\Xi_{BE}} = -kT \frac{\partial}{\partial \epsilon_i} \ln \Xi_{BE}$$

using our trick

$$\bar{n}_i = \frac{e^{-\beta(\epsilon_i - \mu)}}{1 - e^{-\beta(\epsilon_i - \mu)}} = \boxed{\frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}}$$

This is the Bose Einstein distribution function

<u>Notes</u>: * μ must be $\leq \epsilon_i$ for all i otherwise $\bar{n}_i < 0$

for photons we don't pay any price for including more particles so $\mu = 0$

* occupations of different levels are statistically uncorrelated

~~[illegible]~~

* $\overline{\Delta N^2} = 0$ if we do not allow for particles to exchange w/ the reservoir

* $\overline{\Delta n_s^2} = \frac{1}{\beta^2} \frac{\partial^2 \ln \Xi}{\partial \epsilon_s^2}$

* ~~[illegible]~~ Limits are complicated, can't just look at kT need to see how combination of μ & kT behave so that constraints ~~[illegible]~~ for $\bar{N}$ are obeyed

We will return to this

Fermi Dirac Statistics

Same procedure as w/ Bosons we ~~calc~~ calculate Ξ_{FD} so that we do not need to worry about the ~~constr~~ allowed combinations in our sums. The difference is that now each state can only hold 0 or 1 particles

$$\Xi_{FD} = \sum_{n_1=0}^{1}\sum_{n_2=0}^{1}\sum_{n_3=0}^{1}\cdots e^{-\beta(n_1\epsilon_1+n_2\epsilon_2+\cdots)+\beta(\mu n_1+\mu n_2+\cdots)}$$

$$= \prod_{i=1}^{\infty}\left(\sum_{n_i=0}^{1} e^{-\beta(\epsilon_i-\mu)n_i}\right)$$

$$= \prod_{i=1}^{\infty}\left(1+e^{-\beta(\epsilon_i-\mu)}\right)$$

$$\ln \Xi_{FD} = \sum_{i=1}^{\infty}\ln\left(1+e^{-\beta(\epsilon_i-\mu)}\right)$$

$$\bar{N} = kT\frac{\partial(\ln \Xi_{FD})}{\partial\mu} = \sum_{i=1}^{\infty}\frac{e^{-\beta(\epsilon_i-\mu)}}{1+e^{-\beta(\epsilon_i-\mu)}} = \sum_{i=1}^{\infty}\frac{1}{e^{\beta(\epsilon_i-\mu)}+1}$$

$$\bar{E} = \sum_{i=1}^{\infty}\frac{\epsilon_i}{e^{\beta(\epsilon_i-\mu)}+1}$$

Probability for a given microstate characterized by the occupation $n_1, n_2, \ldots$

$$P(n_1,n_2,n_3,\ldots) = \frac{e^{-\beta\sum_i(\epsilon_i-\mu)n_i}}{\Xi_{FD}}$$

$$\bar{n}_i = \sum_{\{n_i\}} n_i\, P(n_1,n_2,n_3,\ldots n_i,\ldots) = \sum_{\{n_i\}} n_i e^{-\beta\sum_i(\epsilon_i-\mu)n_i}$$

$$\bar{n}_i = -kT\frac{\partial(\ln\Xi)}{\partial\epsilon_i} = \frac{e^{-\beta(\epsilon_i-\mu)}}{1+e^{-\beta(\epsilon_i-\mu)}}$$

$$\boxed{\bar{n}_i = \frac{1}{e^{\beta(\epsilon_i-\mu)}+1}}$$ Fermi Distribution function

distinguishable particles

MB distribution

$$\Xi_{MB} = \sum_{r_1}\sum_{r_2}\sum_{r_3} e^{-\beta\left[(\epsilon_{r_1}-\mu) + \epsilon_{r_2}-\mu + \epsilon_{r_3}-\mu + \cdots\right]}$$

$$= \left[\sum_{r_1} e^{-\beta(\epsilon_{r_1}-\mu)}\right]^N$$

$$\overline{n_i} = -\frac{1}{\beta}\frac{\partial \ln \Xi_{MB}}{\partial \epsilon_i} = \frac{N\, e^{-\beta(\epsilon_i-\mu)}}{\sum_{r_1} e^{-\beta(\epsilon_{r_1}-\mu)}}$$

The differences between the three distributions has to do with the way we sum over all the states to obtain the partition function

For Maxwell Boltzman we sum over states r

For Bosons we sum over # of particles in each state

- Photons no restriction on n $\sum_{n=0}^{\infty}$
- particles use our trick of setting $N=\bar{N}$ & using grand canonical ~~dis~~ distribution

For Fermions we sum over the # of particles in each state

- use our trick of setting $N=\bar{N}$ & use grand canonical dist

---

Limits: Do all three distributions agree in the classical limit?

$$\overline{n_i} = \frac{1}{e^{\beta(\epsilon_i-\mu)} \pm 1} \qquad \begin{matrix} + \text{ FD} \\ - \text{ BE}\end{matrix}$$

if gas consists of N particles $\sum_i \overline{n_i} = \sum_i \frac{1}{e^{\beta(\epsilon_i-\mu)} \pm 1} = N$

this allows us to calculate μ. For different β, μ must adjust so that the sum is satisfied.

~~[illegible] β must be very large & [illegible] i.e. $\alpha \gg$~~

To prevent this sum from exceeding N the parameter $\alpha = -\mu\beta$ must become large enough so that each term is sufficiently small $\overline{n_i} \ll 1$

ie. $e^{\beta\epsilon_i + \alpha} \gg 1$ ~~[illegible]~~ only terms w/ $\beta\epsilon_i \ll \alpha$ contribute significantly so as $\beta \to 0$ an increasing # of terms w/ large values of ϵ_i contribute to the sum.

Since $e^{\beta(\epsilon_i-\mu)} \gg 1$

$$\overline{n_i} = e^{-\beta(\epsilon_i-\mu)}$$

$$\sum_i e^{-\beta(\epsilon_i-\mu)} = N$$

$$\overline{n_i} = \frac{e^{-\beta(\epsilon_i-\mu)}\,N}{\sum_i e^{-\beta(\epsilon_i-\mu)}}$$ classical MB result ✓

what about partition functions?

we expect that $\frac{\Xi_{MB}}{N!} = \Xi_{FD,BE}$ in the classical limit

$e^{\beta(\epsilon-\mu)} \gg 1$ or $e^{-\beta(\epsilon-\mu)} \ll 1$

$$\ln\left(1 \pm e^{-\beta(\epsilon-\mu)}\right) = \pm e^{-\beta(\epsilon-\mu)}$$

$$\ln \Xi_{BE} = -\sum_{i=1}^{\infty} \ln\left(1 - e^{-\beta(\epsilon_i-\mu)}\right) \overset{\text{classical limit}}{=} +\sum_{i=1}^{\infty} e^{-\beta(\epsilon_i-\mu)} = N$$ ✓

$$\ln \Xi_{FD} = +\sum_{i=1}^{\infty} \ln\left(1 + e^{-\beta(\epsilon_i-\mu)}\right) \overset{\text{classical limit}}{=} \sum_{i=1}^{\infty} e^{-\beta(\epsilon_i-\mu)} = N$$ ✓

$$\ln \frac{\Xi_{MB}}{N!} = N \ln \underbrace{\sum_{i=1}^{\infty} e^{-\beta(\epsilon-\mu)}}_{N} - N\ln N + N = N$$ ✓

---

low T limit

MB each particle occupies its ground state since $\overline{n_i} \propto e^{-\beta\epsilon_i}$

BE Bose Einstein Condensates (reading material ~~discuss later~~)

FD discuss below:

$$F(\epsilon) \equiv \frac{1}{e^{\beta(\epsilon-\mu)}+1}$$

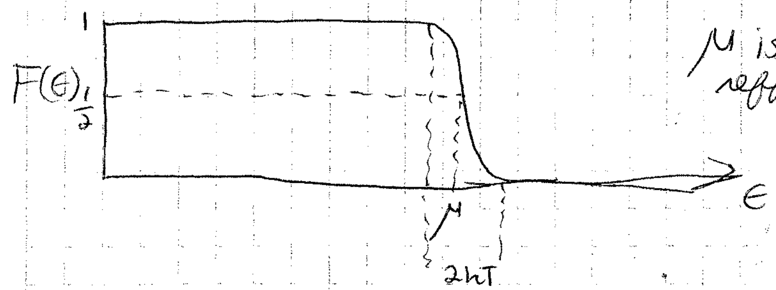
once again μ is a function of β since

$$\sum_i \overline{n_i} = \sum_i \frac{1}{e^{\beta(\epsilon-\mu)}+1} = N$$

if $\beta\mu \gg 1$: 3 cases

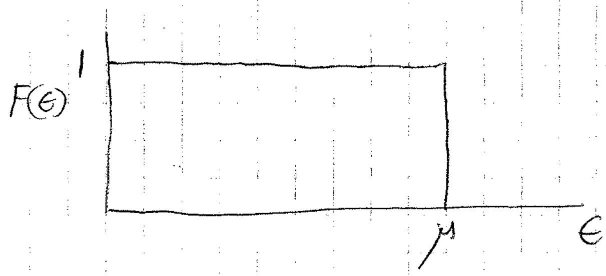
1) $\epsilon \ll \mu$ then $\beta(\epsilon-\mu) \ll 0$ so $F(\epsilon) = 1$
2) $\epsilon \gg \mu$ then $\beta(\epsilon-\mu) \gg 0$ so $F(\epsilon) = e^{\beta(\mu-\epsilon)} \to 0$
3) $\epsilon = \mu$ then $F(\epsilon) = \frac{1}{2}$

at finite T



μ is the chemical potential and is also reffered to as the Fermi energy

at $T=0$

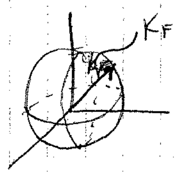


It is useful to calculate $\mu_{T=0}$ to get a feel for physics

$$\epsilon = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} \qquad \mu_0 = \frac{P_F^2}{2m} = \frac{\hbar^2 k_F^2}{2m}$$

all states w/ $k < k_F$ are filled & all states w/ $k > k_F$ are empty

in k space (3-D)



Volume of occupied sphere is $\frac{4}{3}\pi k_F^3$

The question is how many states are there per unit volume of k space?

For PIB Boundary Conditions give $k_x = \frac{2\pi}{L_x} n_x$, $k_y = \frac{2\pi}{L_y} n_y$, $k_z = \frac{2\pi}{L_z}$

# of integers Δn_x for which k_x lies in the range k_x & $k_x + dk_x$ is

$$\Delta n_x = \frac{L_x}{2\pi}\, dk_x$$

The # of translational states $\rho(\vec{k})\, d^3k = \Delta n_x \Delta n_y \Delta n_z = \frac{L_x L_y L_z}{(2\pi)^3} d^3k$

$$= \frac{V}{(2\pi)^3} d^3k$$

Thus there are

$$\frac{V}{(2\pi)^3} \cdot \frac{4}{3}\pi k_F^3 \text{ states } \times 2 \text{ for spin up \& spin down}$$

$$N = 2\frac{V}{(2\pi)^3} \frac{4}{3}\pi k_F^3$$

$$k_F = \left(3\pi^2 \frac{N}{V}\right)^{1/3}$$

$$\lambda_F = \frac{2\pi}{k_F} = \frac{2\pi}{(3\pi^2)^{1/3}}\left(\frac{V}{N}\right)^{1/3}$$ deBroglie wave length

$$\mu_0 = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3}$$

$T_F = \frac{\mu_0}{k_B}$ of Cu is $\approx 80{,}000\,°K$!