Homework 4
Laplace Equation (2)

Ex. 4.1: Conducting sphere with net charge
Calculate the potential at all points in space exterior to a conducting sphere of radius \(a\) placed in a uniform electric field \(E_0\). The sphere has net charge \(q\).

Ex. 4.2: Spherical surface with prescribed potential
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Ex. 4.3: Depolarizing Factor
Consider an infinitely long dielectric cylinder of radius \(a\) and dielectric constant \(\epsilon\). The cylinder is placed in an electric field \(E_0\) directed perpendicular to the cylinder’s axis.

a) Find the electrostatic potential \(\Phi\) at all points outside and inside the cylinder.

b) Find the corresponding electric field \(E\).

c) Show that the field inside the cylinder can be written as a superposition of the applied field and the incremental field produced by the polarization induced on the electric cylinder,

\[ E = E_0 - 4\pi LP, \]

where \(P\) is the polarization in the cylinder and \(L\) is the "depolarizing factor". What value does \(L\) take?

Note: The difference between the actual field \(E\) inside the cylinder and the applied field \(E_0\) is known as the "depolarizing field".

Ex. 4.4: Demagnetizing field
For an arbitrary stationary current density \(j(r)\), the vector potential \(A(r)\) reads

\[ A(r) = \int_V dv' \frac{j(r')}{|r - r'|} \quad (1) \]

where the integral is over the volume \(V\) in which the current density \(j\) is nonzero. (It would be more precise to state that Eq. (1) represents a possible solution of the vector potential, since there are other realizations of the vector potential that correspond to the same magnetic field \(B = \nabla \times A\).)
a) Argue that, away from the currents that give rise to the magnetic field, the vector potential $\mathbf{A}$ corresponding to Eq. (1) above satisfies the Laplace equation

$$\Delta \mathbf{A} = 0.$$ 

b) Consider an infinite cylinder of radius $a$, made of a magnetic material with relative magnetic permeability $\mu$. The cylinder is placed in a magnetic field $\mathbf{B}_0$ directed perpendicular to the cylinder’s axis. Find the vector potential $\mathbf{A}$ and the magnetic field $\mathbf{B}$ at all points inside and outside the cylinder. Show that, for a strongly magnetic material with $\mu \gg 1$ the internal field is essentially twice the applied field $\mathbf{B}_0$.

Hint: Use your answer to Ex. 4.1 a.

c) Now consider the force (per unit length) on the cylinder if the cylinder carries a current $I$. According to the Lorentz force formula, the force (per unit length $d\mathbf{l}$) on the cylinder is

$$d\mathbf{F} = \frac{I}{c} d\mathbf{l} \times \mathbf{B}.$$ 

Which magnetic field do we have to take for $\mathbf{B}$, the internal field you calculated in (b) or the external field $\mathbf{B}_0$? Explain your answer.

**Ex. 4.5: Conducting Cylinder**

Consider a hollow conducting cylinder of radius $a$ and length $L$, of which the two ends are closed by conducting plates insulated electrically from the walls of the cylinder.

a) Calculate the potential $\Phi$ inside the cylinder if the walls of the cylinder and one of the endplates are held at potential $\Phi = 0$, whereas the other endplate is held at potential $-\Phi_0$.

b) Calculate the potential $\Phi$ inside the cylinder if the walls of the cylinder are held at potential $\Phi = 0$, whereas both endplates are held at potential $-\Phi_0$. 

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