Solutions: Homework 6

Ex. 6.1: Superpositions of Waves

(a) First, the complex Poynting vector is defined as

\[ S = \frac{c}{8\pi} E \times B^* \tag{1} \]

such that its real part corresponds to the time-average of the usual Poynting vector \( \langle S \rangle \). For the first two waves, we have

\[ S = \frac{c}{8\pi} \left( e_x E_1^0 + e_x E_2^0 e^{i\alpha} \right) e^{i(kz-\omega t)} \times \left( e_y E_1^0 - e_x E_2^0 e^{-i\alpha} \right) e^{-i(kz-\omega t)} \]

\[ = \frac{c}{8\pi} \left( (E_1^0)^2 + (E_2^0)^2 \right) e_z. \tag{2} \]

In this case, \( S \) is purely real and coincides with \( \langle S \rangle \). Moreover, clearly each wave has complex Poynting vectors \( S_1 = \langle c/(8\pi)(E_1^0)^2 e_z \rangle \) and \( S_2 = \langle c/(8\pi)(E_2^0)^2 e_z \rangle \), so \( S = S_1 + S_2 \). This result is due to the orthogonal polarization of the two waves: \( E_1 \perp E_2 \) and they both propagate in the same direction.

(b) For waves 1 and 3 we have

\[ S = \frac{c}{8\pi} \left( e_x E_1^0 + e_x E_3^0 e^{i\alpha} \right) e^{i(kz-\omega t)} \times \left( e_y E_1^0 + e_y E_3^0 e^{-i\alpha} \right) e^{-i(kz-\omega t)} \]

\[ = \frac{c}{8\pi} \left[ (E_1^0)^2 + (E_3^0)^2 + E_1^0 E_3^0 (e^{i\alpha} + e^{-i\alpha}) \right] e_z, \tag{3} \]

and so

\[ \langle S \rangle = \frac{c}{8\pi} \left[ (E_1^0)^2 + (E_3^0)^2 + 2E_1^0 E_3^0 \cos \alpha \right]. \tag{4} \]

(c) Finally, for waves 1 and 4, noting that the \( e^{i\omega t} \) terms all cancel, we have

\[ S = \frac{c}{8\pi} \left( e_x E_1^0 e^{ikz} + e_x E_4^0 e^{-ikz} \right) \times \left( e_y E_1^0 e^{-ikz} - e_y E_4^0 e^{ikz} \right) \]

\[ = \frac{c}{8\pi} \left[ (E_1^0)^2 - (E_4^0)^2 + (E_1^0)^2 (-e^{2ikz} + e^{2ikz}) \right] e_z \]

\[ = \frac{c}{8\pi} \left[ -2i(E_1^0)^2 \sin(2kz) \right] e_z. \tag{5} \]

Taking the real part for the time averaged Poynting vector, it follows that

\[ \langle S \rangle = 0. \tag{6} \]

Note that the individual Poynting vectors are related by \( S_1 = -S_4 \), due to the relative sign of the wave numbers for waves 1 and 4. In other words, waves 1 and 4 are counter-propagating linearly polarized waves of the same amplitude, so their net Poynting vector is zero.

To find the time dependent energy density, we must consider only the real parts of the waves. Explicitly, we have

\[ E(z,t) = \frac{1}{8\pi} \left[ \Re(E_1 + E_4) \right]^2 + \left[ \Re(B_1 + B_4) \right]^2 \]

\[ = \frac{(E_1^0)^2}{8\pi} \left( \cos(kz - \omega t) + \cos(kz + \omega t) \right)^2 \]

\[ + \left( \cos(kz - \omega t) - \cos(kz + \omega t) \right)^2 \]

\[ = \frac{(E_1^0)^2}{2\pi} \left( \cos^2(kz) \cos^2(\omega t) + \sin^2(kz) \sin^2(\omega t) \right) \]

\[ = \frac{(E_1^0)^2}{4\pi} \left( 1 + \cos(2kz) \cos(2\omega t) \right). \tag{7} \]
The electric and magnetic contributions both clearly oscillate in space and time, and further, since \( \cos^2(x + \pi/2) = \sin^2(x) \), they are \( \pi/2 \) out of phase in both space and time.

**Ex. 6.2: Product Theorem**

Let \( F(t) \) and \( G(t) \) be arbitrary complex functions. Then the product of real parts

\[
\Re(F)\Re(G) = \Re[F\Re(G)] \\
= \Re\left( \frac{F}{2}[G + G^*] \right) \\
= \Re\left( \frac{1}{2}(FG + FG^*) \right),
\]

as required.

**Ex. 6.3: Shielding at Low Frequencies**

Consider a thin-walled cylinder, radius \( a \) and wall thickness \( h \), with axis of symmetry aligned to the \( z \) axis and an external magnetic field \( B = B_0 \cos(\omega t)e_z \). The cylinder has conductivity \( \sigma \). Writing \( B \) as a complex variable, with real parts implied, we have \( B = B_0 e^{i\omega t}e_z \).

Now, using the cylindrical coordinates \( (r, \theta, z) \), by Faraday’s law we have inside the tube

\[
\nabla \times E = -\frac{\partial B_{in}}{\partial t} = -\frac{i\omega B_{in}}{c}e^{i\omega t}e_z, \tag{9}
\]

where \( B_{in} \) is the field inside the tube, and is in general complex valued.

Applying cylindrical symmetry, so that \( E = E(r) \) only, then the induced electric field

\[
E(r) = -\frac{i\omega B_{in}r}{2c}e^{i\omega t}e_\theta. \tag{10}
\]

Further by Ohm’s law, the current density in the cylinder \( J = \sigma E \), so we have

\[
J(r) = -\frac{i\omega B_{in}\sigma r}{2c}e^{i\omega t}e_\theta. \tag{11}
\]

It follows that over a length \( L \) of cylinder, the total current induced in the cylinder is

\[
I = L \int_{a-h}^{a} J(r)dr \\
= -L \frac{i\omega B_{in}\sigma}{4c} [a^2 - (a-h)^2]e^{i\omega t}e_\theta \\
\approx -L \frac{i\omega B_{in}\sigma ah}{2c}e^{i\omega t}e_\theta, \tag{12}
\]

provided \( a \gg h \): i.e. the cylinder is thin-walled.

The field \( B_{i} \) produced by this current is simply that of a solenoid. By Ampère’s law in integral form \( B_i = (4\pi/c)I/L \), so that

\[
B_i = -\frac{2\pi\omega B_{in}\sigma ah}{c^2}e^{i\omega t}e_z, \tag{13}
\]

but the total field in the tube

\[
B_{in} = B_0 + B_i = B_0 - iB_{in} \frac{2\pi\omega\sigma ah}{c^2} = B_0 - iB_{in} \frac{ah}{\delta^2}, \tag{14}
\]

where \( \delta \) is the skin depth. Hence

\[
B_{in} = \frac{B_0}{1 + i(ah/\delta^2)}, \tag{15}
\]

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and then the magnitude of the field inside the tube is
\[ |B_{\text{in}}| = \frac{B_0}{\sqrt{1 + (ah/\delta)^2}}. \]  

(16)

**Ex. 6.4: Radiation Pressure**

a) From equation 5.60 in the book, the radiation pressure for a wave at normal incidence is
\[ p = \frac{1}{c} \langle S \rangle \]  

(17)

When the wave is incident obliquely on a target surface, at angle \( \theta \) from the normal, then a unit area of the *surface* intercepts an area \( \cos \theta \) of the *wave front*. The component of the resulting force normal to the surface brings in a second factor of \( \cos \theta \). And the force is doubled by the reaction from the reflected radiation. Thus, the pressure normal to the perfectly reflecting surface is
\[ p_n(\theta) = 2 \cos^2 \theta p = \frac{2\langle S \rangle}{c} \cos^2 \theta \]  

(18)

There is no tangential force on the patch of area: the reflecting wave undergoes no change of momentum in the tangential direction (the incident wave tends to push the patch forwards, while the reaction to the reflected wave pushes it equally backward).

*Alternative:* The momentum density in an EM field \( g = S/c^2 \). Now, the momentum density
\[ g_i = \frac{\partial P_i}{\partial V} = \frac{1}{c^2} \frac{\partial}{\partial t} \frac{\partial P_i}{\partial A_k}, \]  

(19)

where \( A_k \) is the area normal to the propagation direction, and the partial derivative on the right is the force per unit area. Hence the pressure vector
\[ p = cg. \]  

(20)

Let \( \Delta g \) be the change in momentum density due to reflection. Then by Newton’s 3rd law, the time-averaged pressure exerted on a reflecting surface with normal \( \mathbf{n} \) is
\[ p = c|\mathbf{n} \cdot \langle \Delta g \rangle| = c|\langle \Delta g \rangle| \cos \theta. \]  

(21)

Further, labelling components normal and tangential to the surface, we have
\[ |\Delta g| = | -g_n + g_t - (g_n + g_t)| = |2g_n| = 2|g| \cos \theta. \]  

(22)

Hence the pressure
\[ p = \frac{2\langle S \rangle}{c} \cos^2 \theta. \]  

(23)
b) When the plane wave is incident on a reflecting sphere, the radiation pressure on an area element $da$ produces the force of part (a) in the inward-radial direction.

From symmetry, the only contribution to the net force on the sphere is the component of this radial force in the direction of travel of the incident wave, which brings in yet another factor of $\cos \theta$. Thus

$$F_{\text{reflecting sphere}} = \int p_n(\theta) \cos \theta \ 2\pi R^2 \sin \theta \ d\theta = \frac{4\pi R^2 \langle S \rangle}{c} \int_0^{\pi/2} \cos^3 \theta \sin \theta \ d\theta = \frac{\pi R^2 \langle S \rangle}{c}$$

(24)

c) For a perfectly absorbing sphere, the wave momentum is completely transferred to the sphere. The easiest approach in this case is to observe that the sphere presents the cross-sectional area $\pi R^2$ to the incoming wave, and therefore the total force is simply

$$F_{\text{absorbing sphere}} = p \pi R^2 = \frac{\pi R^2 \langle S \rangle}{c}$$

(25)

which is the same result as the reflecting case. More laboriously, we can paraphrase part (a) to see that the normal force on a surface element $da$ is $dF_n = \cos^2 \theta \ p \ da$ (no factor of 2), and the tangential force is $dF_t = \sin \theta \cos \theta \ p \ da$. Then paraphrasing part (b) the net force on the sphere is

$$\int (\cos \theta dF_n + \sin \theta dF_t) = \frac{\langle S \rangle}{c} \int_0^{\pi/2} (\cos^3 \theta + \sin^2 \theta \cos \theta) 2\pi R^2 \sin \theta d\theta$$

(26)

which agrees with the simpler calculation.

It may seem strange that the same force is obtained for both the reflecting and absorbing cases. The reflecting sphere reflects the incoming radiation in the "backward" sense for polar angles less than $\pi/4$, but in the "forward" sense for angles between $\pi/4$ and $\pi/2$. The net momentum of the reflected radiation turns out to be zero. So the net momentum exchange between wave and sphere is the same in both cases.

(Taken from Hess & Marion Solutions Manual)