Solutions: Homework 9

Ex. 9.1: TM Modes in a Rectangular Waveguide

(a) Components of the electric field tangential to the conducting surface must be zero, so $E_z(x, y)$ must satisfy

$$E_z(0, y) = E_z(a, y) = E_z(x, 0) = E_z(x, b) = 0 .$$

(b) In general the $z$ component of the electric field has form $E_z(x, t) = E_z(x, y) e^{i(k_z - \omega t)}$, so that $E_z(x, y)$ is a solution of the Helmholtz equation

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2 \right) E_z = 0 ,$$

where $k_c^2 = -k^2 + \omega^2/c^2$. Applying the boundary conditions (1), the solution is simply

$$E_z(x, y) = E_z^0 \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) ,$$

for $m, n$ integers, provided that

$$k_c^2 = \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) .$$

We must have $E_z > 0$ for TM modes, and hence $m, n \in \mathbb{Z}^+$.

(c) The dispersion relation is then

$$k(\omega) = \sqrt{\frac{\omega^2}{c^2} - k_c^2} = \sqrt{\frac{\omega^2}{c^2} - \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)} .$$

The cutoff frequency $\omega_c$ is defined by $k(\omega_c) = 0$, so that for the TM$_{mn}$ mode

$$\omega_c = c\pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} .$$

(d) Applying Maxwell’s equations and $B_z = 0$, the transverse electric field components are

$$E_x = \frac{ik}{k_c^2} \frac{\partial}{\partial x} E_z = \frac{ikE_z^0}{k_c^2} \frac{m\pi}{a} \cos(m\pi x/a) \sin(n\pi y/b) ,$$

$$E_y = \frac{ik}{k_c^2} \frac{\partial}{\partial y} E_z = \frac{ikE_z^0}{k_c^2} \frac{n\pi}{b} \sin(m\pi x/a) \cos(n\pi y/b) .$$

Further, the transverse magnetic field components are

$$B_x = -\frac{i\omega}{ck_c^2} \frac{\partial}{\partial y} E_z = \frac{i\omega E_z^0}{ck_c^2} \frac{n\pi}{b} \sin(m\pi x/a) \cos(n\pi y/b) ,$$

$$B_y = -\frac{i\omega}{ck_c^2} \frac{\partial}{\partial x} E_z = \frac{i\omega E_z^0}{ck_c^2} \frac{m\pi}{a} \cos(m\pi x/a) \sin(n\pi y/b) .$$
(e) The time-averaged Poynting vector for the TM$_{mn}$ mode

\[ \langle S \rangle = \frac{c}{8\pi} \text{Re} \left( E \times B^* \right) \]

\[ = \frac{c}{8\pi} \text{Re} \left[ \left( E_x B_y^* - E_y B_x^* \right) e_z + E_z B_y^* e_y - E_z B_x^* e_x \right] \]

\[ = \frac{c}{8\pi} \text{Re} \left[ \left( E_x B_y^* - E_y B_x^* \right) e_z \right], \tag{11} \]

since the other terms are purely imaginary, due to the factors of $i$ in the transverse field components. Hence

\[ \langle S \rangle = \frac{\omega k E_0 z}{32\pi k_c^2} \left[ \left( \frac{m\pi}{a} \right)^2 \cos^2(m\pi x/a) \sin^2(n\pi y/b) + \left( \frac{n\pi}{b} \right)^2 \sin^2(m\pi x/a) \cos^2(n\pi y/b) \right] e_z. \tag{12} \]

Averaging over the cross-section, note

\[ \frac{1}{a} \int_0^a \cos^2 \left( \frac{m\pi x}{a} \right) dx = \frac{1}{b} \int_0^b \cos^2 \left( \frac{n\pi y}{b} \right) dy = \frac{1}{2}, \tag{13} \]

so that the cross-section-and-time-averaged Poynting vector is

\[ \langle \langle S \rangle \rangle = \frac{1}{ab} \int_A dxdy \langle S \rangle \]

\[ = \frac{\omega k E_0^2}{32\pi k_c^2} \left[ \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right] e_z \]

\[ = \frac{\omega k E_0^2}{32\pi} e_z. \tag{14} \]

**Ex. 9.2: Waveguide with Circular Cross Section**

Consider a waveguide with circular cross section, radius $a$. For a TE mode, in cylindrical coordinates $(r, \theta, z)$ we expect $E = E_0(r, \theta) e^{i(kz - \omega t)}$, and so the wave equation becomes the Helmholtz equation

\[ \left( r \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} + k_c^2 r^2 \right) B_z^0(r, \theta) = 0 , \tag{15} \]

where $k_c^2 = \omega^2/c^2 - k^2$, and the boundary condition is

\[ \left. \frac{\partial B_z^0}{\partial r} \right|_{r=a} = 0 . \tag{16} \]

Applying separation of variables as usual, $B_z^0(r, \theta) = B_z^0(r) e^{\pm in\theta}$ and (15) becomes Bessel’s equation

\[ \left( r \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + k_c^2 r^2 - n^2 \right) B_z^0(r, \theta) = 0 . \tag{17} \]

As we saw previously, the general solution is $B_z^0(r) = \sum_{m,n} [A_{mn} J_n(k_{c_m} r) + C_{mn} N_n(k_{c_m} r)]$, but since $B_z^0$ must be regular at $r = 0$ then $C_{mn} = 0$. The general solution is then

\[ B_z^0(r, \theta) = \sum_{m,n} A_{mn} J_n(k_{c_m} r) e^{\pm i(n\theta)} , \tag{18} \]

and the TE$_{mn}$ mode, after taking real parts, is

\[ B_z^0(r, \theta) = A_{mn} J_n(k_{c_m} r) \cos(n\theta) . \tag{19} \]
Note the mode is doubly degenerate, since we may pick the sign of the exponential to be positive or negative, but the real part of $B_z^0$ is the same in either case. We also have the boundary condition for the $TE_{mn}$ mode

$$\frac{d}{dr}J_n(k_m r) = 0.$$  
(20)

By convention $k_{m+1} > k_m$, and hence $k_m a$ is the $m$th local extremum (maximum or minimum) of $J_n$.

By definition, the lowest $TE$ mode is that mode which has smallest $k_c$. It follows that the lowest mode must be a $TE_{1n}$ mode which has the smallest $k_c$, satisfying (20). Or in other words, the lowest mode corresponds to the Bessel function $J_n(u)$ whose first local extremum occurs at the smallest possible $u$. The first local extremum of $J_1(u)$ occurs at $u = 1.841$, which is smaller than the first local extrema of any other Bessel-$J$ function. This can easily be seen from a plot of the first few Bessel functions. Hence, the lowest $TE$ mode is the $TE_{11}$ mode ($n = 1$),

$$B_z^0(r, \theta) = B^0 J_1(k_c r) \cos \theta,$$  
(21)

such that $k_c a = 1.841$ and $B^0 = A_{11}$.

Now, to find the transverse field components of $TE_{11}$, it is convenient to restore the imaginary part of $B_z^0$. The mode is doubly degenerate, i.e. $B_z^0 = B^0 J_1(k_c r) e^{\pm i \theta}$, so choosing the sign in the exponential to be positive, for the magnetic field components we have

$$B_t = \frac{1}{ik_c^2} \nabla B_z^0$$

$$= \frac{k}{ik_c^2} \left( \frac{\partial B_z^0}{\partial r} e_r + \frac{1}{r} \frac{\partial B_z^0}{\partial \theta} e_\theta \right)$$

$$= \frac{kB^0}{k_c^2} \left[ \frac{1}{r} J_1(k_c r) - k_c J_2(k_c r) \right] \sin \theta e_r + \frac{1}{r} J_1(k_c r) \cos \theta e_\theta$$

$$\equiv B_r e_r + B_\theta e_\theta,$$  
(22)

where we have taken the real parts. Further, the transverse electric field satisfies $e_z \times E_t = (\omega/c k) B_t$. Writing $E_t = E_r e_r + E_\theta e_\theta$ and taking the cross product with $e_z$, it follows that

$$E_r e_\theta - E_\theta e_r = \frac{\omega}{ck} (B_r e_r + B_\theta e_\theta),$$  
(23)

so $E_r = (\omega/c k) B_\theta$ and $E_\theta = -(\omega/c k) B_r$. That is,

$$E_t = \frac{\omega B^0}{ck_c^2} \left[ \frac{1}{r} J_1(k_c r) \cos \theta e_r - \left( \frac{1}{r} J_1(k_c r) - k_c J_2(k_c r) \right) \sin \theta e_\theta \right].$$  
(24)

Note that the transverse components for the other degenerate $TE_{11}$ mode are the same up to a change of sign for the $\cos \theta$ terms.

A colour plot of the electric field for one of the degenerate $TE_{11}$ modes is below, with red indicating longer vectors, and blue shorter. Note the electric field is normal to the waveguide surface, indicated by the black circle.